Angles and Polar Coordinates In Real Normed Spaces

VOLKER THÜREY
Rheinstr. 91
28199 Bremen, Germany *

February 16, 2009

MSC-class: 52A10 Keywords: angles, normed space, polar coordinates

Abstract

We try to create a wise definition of 'angle spaces'. Based on an idea of Ivan Singer, we introduce a new concept of an angle in real Banach spaces, which generalizes the euclidean angle in Hilbert spaces. With this angle it is shown that in every two-dimensional subspace of a real Banach space we can describe elements uniquely by polar coordinates.

Contents

1 Introduction

In a real inner product space $(X, <\ ..\ |\ ..\ >)$ it is well-known that the inner product can be expressed by the norm, namely for $\vec{x}, \vec{y} \in X$, $\vec{x} \neq \vec{0} \neq \vec{y}$,

$$< \vec{x} \mid \vec{y} > = \frac{1}{4} \cdot (\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2) = \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] .$$

*T: 49 (0)421/591777 E-Mail: volker@thuerey.de

Furthermore we have for all $\vec{x},\vec{y} \neq \vec{0}$ the euclidean angle

$$\angle_{Euclid}(\vec{x},\vec{y}) := \arccos \frac{< \vec{x} \mid \vec{y} >}{\|\vec{x}\| \cdot \|\vec{y}\|} \;=\; \arccos \left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right) ,$$

which is defined in terms of the norm, too.
In this paper we deal with real topological vector spaces X provided with a continuous map $\|..\| \longrightarrow \mathbb{R}^+ \cup \{0\}$ which is absolute homogeneous, i.e. $\|r \cdot \vec{x}\| = |r| \cdot \|\vec{x}\|$ for $\vec{x} \in X$, $r \in \mathbb{R}$. We call such pairs $(X, \|..\|)$ homogeneously weighted vector spaces (or **hw spaces**). The subset $\mathsf{Z} := \{ \vec{x} \in X \mid \|\vec{x}\| = 0 \} \subset X$ is called the zero-set of $(X, \|..\|)$.
Following the lines of an inner product we define for such spaces a product $< .. \mid .. >_{\spadesuit} : X^2 \longrightarrow \mathbb{R}$, writing for all $\vec{x},\vec{y} \in X$:

$$< \vec{x} \mid \vec{y} >_{\spadesuit} \;:=\; \begin{cases} 0 & \text{for } \|\vec{x}\| \cdot \|\vec{y}\| = 0 \\ \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] & \text{for } \|\vec{x}\| \cdot \|\vec{y}\| \neq 0 \end{cases} .$$

and it is easy to show that such product fulfils the symmetry ($< \vec{x}|\vec{y} >_{\spadesuit} = < \vec{y}|\vec{x} >_{\spadesuit}$), the positive semidefiniteness ($< \vec{x}|\vec{x} >_{\spadesuit} \geq 0$), and the homogenity ($< r \cdot \vec{x}|\vec{y} >_{\spadesuit} = r \cdot < \vec{x}|\vec{y} >_{\spadesuit}$), for $\vec{x},\vec{y} \in X$, $r \in \mathbb{R}$.
For arbitrary **hw spaces** $(X, \|..\|)$, we are able to define for $\vec{x},\vec{y} \in X \backslash \mathsf{Z}$ with $| < \vec{x} \mid \vec{y} >_{\spadesuit} | \leq \|\vec{x}\| \cdot \|\vec{y}\|$ an 'angle', acording to the euclidean angle in inner product spaces.

$$\angle_{Thy}(\vec{x},\vec{y}) := \arccos \frac{< \vec{x} \mid \vec{y} >_{\spadesuit}}{\|\vec{x}\| \cdot \|\vec{y}\|} = \arccos \left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right) .$$

Then we state that in the case of a seminormed space $(X, \|..\|)$, that the triple $(X, \|..\|, < .. \mid .. >_{\spadesuit})$ satisfies the *Cauchy-Schwarz-Bunjakowsky Inequality* or CSB inequality, that means for all $\vec{x},\vec{y} \in X$ we have the inequality $| < \vec{x} \mid \vec{y} >_{\spadesuit} | \leq \|\vec{x}\| \cdot \|\vec{y}\|$. Hence in a real normed vector space $(X, \|..\|)$ the 'Thy angle' $\angle_{Thy}(\vec{x},\vec{y})$ is defined for all $\vec{x},\vec{y} \neq \vec{0}$.
This new 'angle' has eight nice properties, which are known from the euclidean angle in inner product spaces and corresponds with the euclidean angle in the case that $(X, \|..\|)$ already is an inner product space.
Let $(X, \|..\|)$ be a real normed space, let $\dim(X) > 1$, let $\vec{x},\vec{y} \neq \vec{0}$. Then we have that

- $\angle_{Thy}$ is a continuous surjective function from $[X \backslash \{\vec{0}\}]^2$ to $[0, \pi]$.
- $\angle_{Thy}(\vec{x},\vec{x}) = 0$.
- $\angle_{Thy}(-\vec{x},\vec{x}) = \pi$.
- $\angle_{Thy}(\vec{x},\vec{y}) = \angle_{Thy}(\vec{y},\vec{x})$.
- For all $r, s > 0$, we have $\angle_{Thy}(r \cdot \vec{x}, s \cdot \vec{y}) = \angle_{Thy}(\vec{x},\vec{y})$.
- $\angle_{Thy}(-\vec{x},-\vec{y}) = \angle_{Thy}(\vec{x},\vec{y})$.
- $\angle_{Thy}(\vec{x},\vec{y}) + \angle_{Thy}(-\vec{x},\vec{y}) = \pi$,

which are all easy to prove. Moreover, the 'Thy angle' has the following important property, which is the main content of this paper, and which is not so easy to prove.
Theorem
For any two linear independent vectors $\vec{x},\vec{y}$, there is a decreasing homeomorphism

$$\Theta : \mathbb{R} \longrightarrow (0, \pi), \quad t \mapsto \angle_{Thy}(\vec{x}, \vec{y} + t \cdot \vec{x}) \; .$$

This theorem is proved in the section **On the Existence of Polar Coordinates**. We use the help of a paper of Charles Dimminie, Edward Andalafte, and Raymond Freese [2].
Furthermore, we work with two interesting facts from the usual two-dimensional euclidean geometry. We do not state them here in the introduction, because the most difficult part is to write them down, rather than to prove them.
At the end, two open questions about **hw spaces** $(X, \|..\|)$ which have a non convex unit ball are described. One possibility to avoid these problems is to replace this homogeneous weight $\|..\|$ by another, using the convex hull of the unit ball. We define a useful generalization of the introduced 'Thy angle', maintaining all its good properties.

2 General Definitions

Let $X = (X, \tau)$ be an arbitrary real topological vector space, that means that the real vector space X is provided with a topology τ such that the addition of two vectors and the multiplication with real numbers are continuous. Further let $\|..\|$ denote a *positive functional* on X, that means that $\|..\|: \ X \longrightarrow \mathbb{R}^+ \cup \{0\}$ is continuous, $\mathbb{R}^+ \cup \{0\}$ carries the usual euclidean topology.
We consider some conditions.
(1) For all $r \in \mathbb{R}$ and all $\vec{x} \in X$ we have: $\|r \cdot \vec{x}\| = |r| \cdot \|\vec{x}\|$ (" absolute homogenity"),
(2) $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$ ("positive definiteness"),
(3) For all $\vec{x}, \vec{y} \in X$ hold $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ ("triangle inequality"),
(4) For all $\vec{x}, \vec{y} \in X$ hold $\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2 \cdot [\ \|\vec{x}\|^2 + \|\vec{y}\|^2\ ]$ ("parallelogram identity"),

If $\|..\|$ fulfils (1) then we call $\|..\|$ a *homogeneous weight* on X,
if $\|..\|$ fulfils (1), (3) then $\|..\|$ is called a *seminorm* on X, and
if $\|..\|$ fulfils (1), (2) and (3) then $\|..\|$ is called a *norm* on X, and
if $\|..\|$ fulfils (1), (2), (3) and (4) then the pair $(X, \|..\|)$ is called an *inner product space*.

Acording to this cases we call the pair $(X, \|..\|)$ a *homogeneously weighted vector space* (or **hw space**), a *seminormed vector space*, a *normed vector space*, or an *inner product space* (or **IP space**), respectively.
Now let $< ..|.. >: \ X^2 \longrightarrow \mathbb{R}$, let $< ..|.. >$ be continuous as a map from the product space $X \times X$ to the euclidean space $\mathbb{R}$. We consider some conditions:
$\overline{(1)}$ For all $r \in \mathbb{R}$ and all $\vec{x}, \vec{y} \in X$ hold $< r \cdot \vec{x} \mid \vec{y} >= r \cdot < \vec{x} \mid \vec{y} >$ ("homogenity"),
$\overline{(2)}$ For all $\vec{x}, \vec{y} \in X$ hold $< \vec{x} \mid \vec{y} > = < \vec{y} \mid \vec{x} >$ ("symmetry"),
$\overline{(3)}$ For all $\vec{x} \in X$ we have: $< \vec{x} \mid \vec{x} > \ \geq \ 0$ ("positive semidefiniteness"),
$\overline{(4)}$ $< \vec{x} \mid \vec{x} > = 0$ if and only if $\vec{x} = \vec{0}$ ("definiteness"),
$\overline{(5)}$ For all $\vec{x}, \vec{y}, \vec{z} \in X$ hold $< \vec{x} \mid \vec{y} + \vec{z} > = < \vec{x} \mid \vec{y} > + < \vec{x} \mid \vec{z} >$ ("linearity in the second component").

If $< .. \mid .. >$ fulfils $\overline{(1)}, \overline{(2)}, \overline{(3)}$, then we call $< .. \mid .. >$ a *homogeneous product* on X,
if $< .. \mid .. >$ fulfils $\overline{(1)}, \overline{(2)}, \overline{(3)}, \overline{(4)}, \overline{(5)}$, then $< .. \mid .. >$ is called an *inner product* on X.

Acording to this cases we call the pair $(X, < .. \mid .. >)$ a *homogeneous product vector space*, or an *inner product space* (or **IP space**), respectively.

Remark 1. We use the term '**IP space**' twice, but both definitions coincide: It is well-known that a norm is based on an inner product if and only if the parallelogram identity holds.

Let $\|..\|$ be denote a positive functional on X. Then define the two closed subsets of X:
$\mathbf{S} := \mathbf{S}_{(X, \|..\|)} := \{\, \vec{x} \in X \mid \|\vec{x}\| = 1 \,\}$, the *unit sphere* of X,
$\mathbf{B} := \mathbf{B}_{(X, \|..\|)} := \{\, \vec{x} \in X \mid \|\vec{x}\| \leq 1 \,\}$, the *unit ball* of X .

Now assume that the real vector space X is provided with a positive functional $\|..\|$ and a product $< .. \mid .. >$. Then the triple $(X, \|..\|, < .. \mid .. >)$ satisfies the *Cauchy-Schwarz-Bunjakowsky Inequality* or CSB inequality $\Longleftrightarrow$ for all $\vec{x}, \vec{y} \in X$ we have the inequality $\mid < \vec{x} \mid \vec{y} > \mid \ \leq \ \|\vec{x}\| \cdot \|\vec{y}\|$.

Assume that the pair $(X, \|..\|)$ is a homogeneously weighted vector space (or **hw space**). Then define for every vector $\vec{v}$ with $\|\vec{v}\| \neq 0$ the vector $\mathsf{sign}(\vec{v}) := \frac{1}{\|\vec{v}\|} \cdot \vec{v}$, thus $\mathsf{sign}(\vec{v})$ is the projection of $\vec{v}$ into the unit sphere $\mathbf{S}_{(X,\|..\|)}$.

Let A be an arbitrary subset of a linear real vector space X . Let A have the property that for arbitrary $\vec{x}, \vec{y} \in A$ and for every $0 \leq t \leq 1$ we have $t \cdot \vec{x} + (1-t) \cdot \vec{y} \in A$. Such a set A is called *convex*. The unit ball $\mathbf{B}$ in a seminormed space is convex because of the triangle inequality.

A convex set A in a linear topological vector space $X = (X, \tau)$ is called *strictly convex* if and only if for all $\vec{x}, \vec{y} \in A$ and for every $0 < t < 1$ holds that $t \cdot \vec{x} + (1-t) \cdot \vec{y} \in interior(A)$.

Let A be an arbitrary subset of a real vector space X. Then we define the *convex hull* of A,

$$conv(A) := \bigcup \{ \sum_{i=1}^{n} t_i \cdot \vec{x_i} \mid n \in \mathbb{N},\ t_i \in [0,1] \ \text{ and } \ \vec{x_i} \in A \ \text{ for } i = 1, ..., n \ , \ \text{and } \sum_{i=1}^{n} t_i = 1 \} \ ,$$

which is the smallest convex set that contains A.

Let the pair $(X, \|..\|)$ be a homogeneously weighted vector space (or **hw space**), with the unit ball $\mathbf{B}$ of X . Let $\|..\|_{|conv(\mathbf{B})}$ be the Minkowski Functional of $conv(\mathbf{B})$ in X, that means for all $\vec{x} \in X$ that $\|\vec{x}\|_{|conv(\mathbf{B})} \ := \ \inf \{ r > 0 \mid \frac{1}{r} \cdot \vec{x} \in conv(\mathbf{B}) \}$. Hence $\|\vec{x}\|_{|conv(\mathbf{B})} \ \leq \ \|\vec{x}\|$. Note that for a **hw space** $(X, \|..\|)$, the pair $(X, \|..\|_{|conv(\mathbf{B})})$ is a seminormed vector space. Then we call $\|..\|$, or the pair $(X, \|..\|)$, respectively, *normable* if and only if the pair $(X, \|..\|_{|conv(\mathbf{B})})$ is a normed vector space.

Let $(X, \|..\|)$ be a real **hw space** . The subset Z of X , $\mathsf{Z} := \{ \vec{x} \in X \mid \|\vec{x}\| = 0 \}$ is called the *zero-set* of $(X, \|..\|)$.

3 On Angle Spaces

In the usual euclidean plane angles are considered for more than 2000 years. With the idea of 'metrics' and 'norms' others than the euclidean one the idea came to have also orthogonality and angles in metric and normed spaces, respectively. The first attempt to define a concept of generalized 'angles' on metric spaces was made by Menger [1],p. 749 . Since then a few ideas have been developed, see the references [2], [3], [4], [5], [8], [9], [10], [11], [12]. In this paper we focus our intention on real normed spaces as a generalizitation of real inner product spaces. Let $(X, < .. \mid .. >)$ be an **IP space**, and let $\|..\|$ be the associated norm, $\|\vec{x}\| := \sqrt{< \vec{x} | \vec{x} >}$, then the triple $(X, \|..\|, < .. \mid .. >)$ fulfils the CSB inequality, and we have for all $\vec{x}, \vec{y} \neq \vec{0}$ the well-known euclidean angle $\angle_{Euclid}(\vec{x}, \vec{y}) := \arccos \frac{<\vec{x} \mid \vec{y}>}{\|\vec{x}\| \cdot \|\vec{y}\|}$ with all its nice properties.

Now we want to create a useful definition of an 'angle space'. Of course, if we think of angles as we used them in **IP spaces**, we wish to get all the properties which are known from these angles. But we have to avoid extremal positions; that means, if we demand too much of the properties of the known euclidean angle, we only can expect to get **IP spaces** as 'angle spaces', see [3], [4]. On the other hand, if we request none, then we will get a lot of 'angle spaces', but without any interesting characteristics. Thus we have to find the golden mean. So let's try:

Definition 1. Let $(X, \|..\|)$ be a real **hw space** . Let $\mathsf{Z} := \{ \vec{x} \in X \mid \|\vec{x}\| = 0 \}$ be the 'zero-set'. We call the triple $(X, \|..\|, \angle_X)$ an *angle space* if and only if the following conditions (`An` 1),(`An` 2),(`An` 3),(`An` 4),(`An` 5) are satisfied.

- (An 1) $\angle_X$ is a continuous function from $[X \backslash \mathsf{Z}]^2$ in the interval $[0, \pi]$.
- (An 2) For all $\vec{x} \in X \backslash \mathsf{Z}$ we have $\angle_X(\vec{x}, \vec{x}) = 0$.
- (An 3) For all $\vec{x} \in X \backslash \mathsf{Z}$ we have $\angle_X(-\vec{x}, \vec{x}) = \pi$.
- (An 4) For all $\vec{x}, \vec{y} \in X \backslash \mathsf{Z}$ we have $\angle_X(\vec{x}, \vec{y}) = \angle_X(\vec{y}, \vec{x})$.
- (An 5) For all $\vec{x}, \vec{y} \in X \backslash \mathsf{Z}$ and for all $r, s > 0$ we have $\angle_X(r \cdot \vec{x}, s \cdot \vec{y}) = \angle_X(\vec{x}, \vec{y})$.

Furthermore we write down some more properties of such conditions which seems to us 'desire-able', but 'not absolutely necessary'.

- (An 6) For all $\vec{x}, \vec{y} \in X \backslash \mathsf{Z}$ we have $\angle_X(-\vec{x}, -\vec{y}) = \angle_X(\vec{x}, \vec{y})$.
- (An 7) For all $\vec{x}, \vec{y} \in X \backslash \mathsf{Z}$ we have $\angle_X(\vec{x}, \vec{y}) + \angle_X(-\vec{x}, \vec{y}) = \pi$.
- (An 8) For all $\vec{x}, \vec{y},\ \vec{x} + \vec{y} \in X \backslash \mathsf{Z}$ we have
$$\angle_X(\vec{x}, \vec{x} + \vec{y}) + \angle_X(\vec{x} + \vec{y}, \vec{y}) = \angle_X(\vec{x}, \vec{y}) \ .$$
- (An 9) For $\vec{x}, \vec{y},\ \vec{x} - \vec{y} \in X \backslash \mathsf{Z}$ we have
$$\angle_X(\vec{x}, \vec{y}) + \angle_X(-\vec{x}, \vec{y} - \vec{x}) + \angle_X(-\vec{y}, \vec{x} - \vec{y}) = \pi \ .$$
- (An 10) For all $\vec{x}, \vec{y},\ \vec{x} - \vec{y} \in X \backslash \mathsf{Z}$ we have
$$\angle_X(\vec{y}, \vec{y} - \vec{x}) + \angle_X(\vec{x}, \vec{x} - \vec{y}) = \angle_X(-\vec{x}, \vec{y}) \ .$$
- (An 11) For any two linear independent vectors $\vec{x}, \vec{y} \in X \backslash \mathsf{Z}$, we have a decreasing homeomorphism $\Theta : \mathbb{R} \longrightarrow (0, \pi), \quad t \mapsto \angle_X(\vec{x}, \vec{y} + t \cdot \vec{x})$.

Remark 2. We add another demand to the above conditions. If we construct an angle $\angle_Y$ for every element $(Y, \|..\|)$ of a class K , and if $\{(X, \|..\|) | (X, \|..\|)$ is an **IP space**$\} \subset \mathsf{K}$, then for every **IP space** $(Y, \|..\|)$ should hold that $\angle_Y = \angle_{Euclid}$.

4 The Thy Angle

Now imagine that the real vector space X is provided with a positive functional $\|..\|$ and a product $< .. \,|\, .. >$. Assume two elements $\vec{x}, \vec{y} \in X,\ \|\vec{x}\| \cdot \|\vec{y}\| \neq 0$, and the property that $| < \vec{x} | \vec{y} > | \ \leq \ \|\vec{x}\| \cdot \|\vec{y}\|$. Then we can define an angle between these two elements, $\angle(\vec{x}, \vec{y}) := \arccos \frac{<\vec{x}\,|\,\vec{y}>}{\|\vec{x}\| \cdot \|\vec{y}\|}$. If the triple $(X, \|..\|, < .. \,|\, .. >)$ satisfies the *Cauchy-Schwarz-Bunjakowsky Inequality* or CSB inequality, then we are able to define for all $\vec{x}, \vec{y} \in X,\ \|\vec{x}\| \cdot \|\vec{y}\| \neq 0$, this angle $\angle(\vec{x}, \vec{y}) := \arccos \frac{<\vec{x}\,|\,\vec{y}>}{\|\vec{x}\| \cdot \|\vec{y}\|} \in [0, \pi]$.

Let the pair $(X, \|..\|\ )$ be a **hw space**, thus $(X, \|..\|\ )$ fulfils (1), the absolute homogenity. We define a product $< .. \,|\, .. >_\spadesuit$ on X . Let for all $\vec{x}, \vec{y} \in X$:

$$< \vec{x} \,|\, \vec{y} >_\spadesuit \ := \ \begin{cases} 0 & \text{for} \quad \|\vec{x}\| \cdot \|\vec{y}\| = 0 \\ \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\ \left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \ - \ \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \ \right] & \text{for} \quad \|\vec{x}\| \cdot \|\vec{y}\| \neq 0 \end{cases}$$

(Note that, in the case that $(X, \|..\|\ )$ is already an **IP space**, this definition corresponds with the usual definition of the inner product.) We have $< .. \,|\, .. >_\spadesuit : \ X^2 \longrightarrow \mathbb{R}$, and the properties $\overline{(2)}$ (symmetry) and $\overline{(3)}$ (positive semidefiniteness) are rather trivial. Clearly,

$\|\vec{x}\| = \sqrt{< \vec{x} \mid \vec{x} >_\spadesuit}$ for all $\vec{x} \in X$. We show $\overline{(1)}$, the homogenity. For a real number $r > 0$ it holds that $< r \cdot \vec{x} \mid \vec{y} >_\spadesuit = r \cdot < \vec{x} \mid \vec{y} >_\spadesuit$, because $(X, \|..\|)$ satisfies (1).
Now we prove $< -\vec{x} \mid \vec{y} >_\spadesuit = - < \vec{x} \mid \vec{y} >_\spadesuit$. Let $\|\vec{x}\| \cdot \|\vec{y}\| \neq 0$. We have

$$- < \vec{x} \mid \vec{y} >_\spadesuit \;=\; -\frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] ,$$

$$\text{and} \quad < -\vec{x} \mid \vec{y} >_\spadesuit \;=\; \frac{1}{4} \cdot \|-\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{-\vec{x}}{\|-\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{-\vec{x}}{\|-\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right]$$

$$= \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{y}}{\|\vec{y}\|} - \frac{\vec{x}}{\|\vec{x}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] ,$$

hence $< -\vec{x} \mid \vec{y} >_\spadesuit = - < \vec{x} \mid \vec{y} >_\spadesuit$. Then easily follows also for every real number $r < 0$ that $< r \cdot \vec{x} \mid \vec{y} >_\spadesuit = r \cdot < \vec{x} \mid \vec{y} >_\spadesuit$, and the homogenity $\overline{(1)}$ has been proved, hence the pair $(X, < ..|.. >_\spadesuit)$ is a homogeneous product vector space.

Definition 2. For all **hw spaces** $(X, \|..\|)$ for all $\vec{x}, \vec{y} \in X \backslash \mathsf{Z}$ (that means $\|\vec{x}\| \cdot \|\vec{y}\| \neq 0$) with $| < \vec{x} \,|\, \vec{y} >_\spadesuit | \leq \|\vec{x}\| \cdot \|\vec{y}\|$ we define the *Thy angle* (which is a modification of the angle discussed in [2], but there the authors implicitly assume the parallelogram identity. See also [8].)
$\angle_{Thy}(\vec{x}, \vec{y}) := \arccos \frac{< \vec{x} \,|\, \vec{y} >_\spadesuit}{\|\vec{x}\| \cdot \|\vec{y}\|} = \arccos \left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right)$.

Proposition 1. (a) If $(X, \|..\|)$ is a real seminormed vector space, then the triple $(X, \|..\|, < .. \,|\, .. >_\spadesuit)$ fulfils the CSB inequality, hence the 'Thy angle' $\angle_{Thy}(\vec{x}, \vec{y})$ is defined for all $\vec{x}, \vec{y}$ with $\|\vec{x}\| \cdot \|\vec{y}\| \neq 0$.
(b) If $(X, \|..\|)$ is a real seminormed vector space, then the triple $(X, \|..\|, \angle_{Thy})$ fulfils all the above demands (An 1), (An 2), (An 3), (An 4), (An 5) . Hence $(X, \|..\|, \angle_{Thy})$ is an angle space.
(c) If $(X, \|..\|)$ is a real seminormed vector space, then the triple $(X, \|..\|, \angle_{Thy})$ fulfils (An 6) and (An 7) .
(d) If $(X, < .. \,|\, .. >_{IP})$ is an **IP space**, then the triple $(X, \|..\|, < .. \,|\, .. >_{IP})$ fulfils the CSB inequality and we have for all $\vec{x}, \vec{y} \neq \vec{0}$ that $\angle_{Thy}(\vec{x}, \vec{y}) = \angle_{Euclid}(\vec{x}, \vec{y})$.
(e) If $(X, \|..\|)$ is a real normed vector space, then the triple $(X, \|..\|, \angle_{Thy})$ generally does not fulfil (An 8), (An 9), (An 10) .
(f) If $(X, \|..\|)$ is a real seminormed vector space, then the triple $(X, \|..\|, \angle_{Thy})$ generally does not fulfil (An 8), (An 9), (An 10), (An 11) .

Proof. (a) If $(X, \|..\|)$ is a real seminormed vector space, then because of the triangle inequality and $\left\| \frac{\vec{x}}{\|\vec{x}\|} \right\| = 1$ we get that $|< \vec{x} \,|\, \vec{y} >_\spadesuit| = \left| \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right|$
$\leq \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \max \left\{ \left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 , \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right\} \leq \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot 2^2 = \|\vec{x}\| \cdot \|\vec{y}\|$.
(b) Rather trivial if you use (a) and the fact that $\|..\|$ is homogeneous.
(c) (An 6) is trivial because $< \,|\, >_\spadesuit$ is homogeneous, and (An 7) is easy if you know that $\arccos(r) + \arccos(-r) = \pi$.
(d) If ($(X, < .. \,|\, .. >_{IP})$ is an **IP space** with the associated norm $\|\vec{x}\| := \sqrt{< \vec{x} \,|\vec{x} >_{IP}}$, then, because of $< .. \,|\, .. >_\spadesuit = < .. \,|\, .. >_{IP}$, we have $\angle_{Thy}(\vec{x}, \vec{y}) = \angle_{Euclid}(\vec{x}, \vec{y})$.
(e) We need counterexamples. Recall the pairs $(\mathbb{R}^2, \|..\|_p)$, with the *Hölder weights* $\|..\|_p$, $p > 0$, we define that $\|(x_1, x_2)\|_p := \sqrt[p]{|x_1|^p + |x_2|^p}$. The pairs $(\mathbb{R}^2, \|..\|_p)$ are normed

spaces if and only if $p \geq 1$. For $p = 2$ we get the usual euclidean norm. So let us take, for instance, $p = 1$, because it is easy to calculate with.
Let $\vec{x} := (1,0)$, $\vec{y} := (0,1)$, both vectors have the $\|..\|_1$-norm 1 . Then we have

$$\begin{aligned}
\angle_{Thy}(\vec{x},\vec{y}) &= \arccos\left(\frac{1}{4}\cdot\left[\ \left\|\frac{\vec{x}}{\|\vec{x}\|_1}+\frac{\vec{y}}{\|\vec{y}\|_1}\right\|_1^2 - \left\|\frac{\vec{x}}{\|\vec{x}\|_1}-\frac{\vec{y}}{\|\vec{y}\|_1}\right\|_1^2\ \right]\right)\\
&= \arccos\left(\frac{1}{4}\cdot\left[\ \|(1,0)+(0,1)\|_1^2 - \|(1,0)-(0,1)\|_1^2\ \right]\right)\\
&= \arccos\left(\frac{1}{4}\cdot[\ 4 - 4\ ]\right) = \arccos(0) = \pi/2 = 90\,\mathrm{deg}\,.
\end{aligned}$$

And
$$\begin{aligned}
\angle_{Thy}(\vec{x},\vec{x}+\vec{y}) &= \arccos\left(\frac{1}{4}\cdot\left[\ \left\|(1,0)+\frac{1}{2}\cdot(1,1)\right\|_1^2 - \left\|(1,0)-\frac{1}{2}\cdot(1,1)\right\|_1^2\ \right]\right)\\
&= \arccos\left(\frac{1}{4}\cdot\left[\ (2)^2 - (1)^2\ \right]\right) = \arccos\left(\frac{3}{4}\right) \approx 41.41\,\mathrm{deg}\,.
\end{aligned}$$

With similar calculations, we get $\angle_{Thy}(\vec{x}+\vec{y},\vec{y}) = \arccos\left(\frac{3}{4}\right)$, hence $\angle_{Thy}(\vec{x},\vec{x}+\vec{y}) + \angle_{Thy}(\vec{x}+\vec{y},\vec{y}) \neq \angle_{Thy}(\vec{x},\vec{y})$, and that contradicts (An 8).
The condition (An 9) means that the sum of the inner angles of a triangle is π.
We can use the same example of the normed space $(\mathbb{R}^2,\|..\|_1)$ with unit vectors $\vec{x} := (1,0)$, and $\vec{y} := (0,1)$. Again we get
$\angle_{Thy}(\vec{x},\vec{y}) = \pi/2$, $\angle_{Thy}(-\vec{x},\vec{y}-\vec{x}) = \angle_{Thy}(-\vec{y},\vec{x}-\vec{y}) = \arccos(\frac{3}{4})$, hence
$\angle_{Thy}(\vec{x},\vec{y}) + \angle_{Thy}(-\vec{x},\vec{y}-\vec{x}) + \angle_{Thy}(-\vec{y},\vec{x}-\vec{y}) < \pi$, hence (An 9) is not fulfiled.
For the condition (An 10) we use the same space and the same vectors $\vec{x} := (1,0)$, and $\vec{y} := (0,1)$. We get $\angle_{Thy}(-\vec{x},\vec{y}) = \pi/2$, $\angle_{Thy}(\vec{y},\vec{y}-\vec{x}) = \angle_{Thy}(\vec{x},\vec{x}-\vec{y}) = \arccos(\frac{3}{4})$, hence (An 10) is not fulfiled.
(f) We use the same example as in (e) to prove that (An 8), (An 9), (An 10) generally is not fulfiled. Or we can change it, so that it is no more a normed space. Take the pair ($\mathbb{R}^3,\|..\|_{\hat{1}}$) , with $\|..\|_{\hat{1}}(x,y,z) := |x|+|y|$. Obviously, it is a seminormed, but not a normed space, with does not fulfil (An 8), (An 9), (An 10).
Here is a further example. Let $(X,\|..\|) := (\mathbb{R}^2,\|..\|)$ be the seminormed space with the seminorm $\|(x_1,x_2)\| := |x_1|$. Hence $\mathsf{Z} = \{(0,x_2) \mid x_2 \in \mathbb{R}\}$ is the zero-set. We get only two angles, for all $\vec{x},\vec{y} \in \mathbb{R}^2\backslash\mathsf{Z}$ hold that $\angle_{Thy}(\vec{x},\vec{y}) \in \{0,\pi\}$. Then (An 11) is not satisfied: Take $\vec{x} := (1,0)$, $\vec{y} := (1,1)$, then for all $t \in \mathbb{R}\backslash\{-1\}$ we have that for $t > -1$, $\angle_{Thy}(\vec{x},\vec{y}+t\cdot\vec{x}) = 0$, and for $t < -1$, $\angle_{Thy}(\vec{x},\vec{y}+t\cdot\vec{x}) = \pi$. The calculations for this are easy. □

Another interesting non-trivial example is the following.
Let $(X,\|..\|) := (\mathbb{R}^2,\|..\|)$ be a **hw space** with the unit sphere $\mathbf{S} := \{\ \vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\| = 1\ \} := \{(x_1,x_2) \in \mathbb{R}^2 \mid |x_2|\cdot|x_1| = 1\ \}$. Hence $\mathsf{Z} = \{(0,x_2) \mid x_2 \in \mathbb{R}\} \cup \{(x_1,0) \mid x_1 \in \mathbb{R}\}$ is the zero-set. This space fulfils the CSB inequality, and $(\mathbb{R}^2,\|..\|,\angle_{Thy})$ satisfies (An 1), (An 2), (An 3), (An 4), (An 5), (An 6), (An 7) . Hence ($\mathbb{R}^2,\|..\|,\angle_{Thy}$) is an angle space, which is not a seminormed space (the unit ball is not convex). We have $\angle_{Thy}(\vec{x},\vec{y}) \in \{0,\pi/2,\pi\}$ for all $\vec{x},\vec{y} \in \mathbb{R}^2\backslash\mathsf{Z}$.

5 On the Existence of Polar Coordinates

Theorem 1. Assume that $(\ X, \|..\|\ )$ is a real normed vector space. Then the Thy angle $\angle_{Thy}$ satisfies $(\texttt{An}\ 11)$. In other words, for a linear independent subset $\{\vec{x}, \vec{y}\} \subset X$ we get a decreasing homeomorphism $\Theta : \mathbb{R} \longrightarrow (0, \pi), \quad t \mapsto \angle_{Thy}(\vec{x}, \vec{y} + t \cdot \vec{x})$.

The above theorem is the main result of this paper. Before we start the lengthy proof (finished on page 17) we formulate two comments.

Remark 3. The theorem remains true when we formulate it more general, but less beautiful: ' Assume that $(\ X, \|..\|\ )$ is a real seminormed vector space. Assume that a linear independent subset $\{\vec{x}, \vec{y}\} \subset X$ generates the two-dimensional subspace $U \subset X$, assume $\mathsf{Z} \cap U = \{\vec{0}\}$. Then we have a decreasing homeomorphism $\Theta : \mathbb{R} \longrightarrow (0, \pi), \quad t \mapsto \angle_{Thy}(\vec{x}, \vec{y} + t \cdot \vec{x})$ ' .

Corollary 1. With the above theorem we can describe elements of a two-dimensional real normed vector space $(X, \|..\|)$ by polar coordinates. If we fix a basis $\{\vec{b_1}, \vec{b_2}\}$, then every $\vec{x} = r_1 \cdot \vec{b_1} + r_2 \cdot \vec{b_2} \in X$ is uniquely defined by its norm $\|\vec{x}\|$ and its angle $\angle_X(\vec{x}) := \angle_{Thy}(\vec{x}, \vec{b_1})$ if and only if $r_2 > 0$ and $\angle_X(\vec{x}) := -\angle_{Thy}(\vec{x}, \vec{b_1})$ if and only if $r_2 < 0$. This concept easily can be extended to finite dimensional real normed vector spaces.

Now we start the proof of the above theorem. The reader should have a copy of the remarkable paper of Charles Dimminie, Edward Andalafte, and Raymond Freese [2], because we need some propositions from that paper, which we write down without the proofs.
The central idea of the proof is, assuming that the map Θ is not injective contradicts the convexity of the unit ball $\mathbf{B}$ of $(X, \|..\|)$.

Proof. (of **Theorem 1**)
Assume that $(\ X, \|..\|\ )$ is a real normed vector space. (Hence $\mathsf{Z} = \{\vec{0}\}$) . Assume that a linear independent subset $\{\vec{x}, \vec{y}\} \subset X$ generates the two-dimensional subspace U. We consider the map $\Theta : \mathbb{R} \longrightarrow [0, \pi]$. For convenience, we define some abbreviations. Let $h_+, h_- : \mathbb{R} \longrightarrow [0, 2], \quad h_+(t) := \left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}+t\cdot\vec{x}}{\|\vec{y}+t\cdot\vec{x}\|} \right\|, \quad h_-(t) := \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}+t\cdot\vec{x}}{\|\vec{y}+t\cdot\vec{x}\|} \right\|$. We have

$$\begin{aligned}
\Theta(t) := \angle_{Thy}(\vec{x}, \vec{y} + t \cdot \vec{x}) &= \arccos\left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y} + t \cdot \vec{x}}{\|\vec{y} + t \cdot \vec{x}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y} + t \cdot \vec{x}}{\|\vec{y} + t \cdot \vec{x}\|} \right\|^2 \right] \right) \\
&= \arccos\left(\frac{1}{4} \cdot \left[[h_+(t)]^2 - [h_-(t)]^2 \right] \right) ,
\end{aligned}$$

and let $\Theta(-\infty) := \pi, \quad \Theta(+\infty) := 0$.

Lemma 1. We have that
$\lim_{t\to-\infty} h_+(t) = \lim_{t\to+\infty} h_-(t) = 0$, and $\lim_{t\to+\infty} h_+(t) = \lim_{t\to-\infty} h_-(t) = 2$.

Proof. See [8],p.38 , or [2],p.199 . □

Lemma 2. We have that the map $\Theta : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0, \pi]$ is continuous and surjective.

Proof. By the previous lemma $\lim_{t\to-\infty} \Theta(t) = \pi$ and $\lim_{t\to+\infty} \Theta(t) = 0$, and the norm $\|..\|$ is continuous, hence Θ is continuous and the image of Θ is $[0, \pi]$. □

We still have to prove the injectivity of Θ , the difficult part of the proof. In the following we need the notion of a strictly convex set, that is a subset A of a linear topological vector space such that $t \cdot \vec{x} + (1 - t) \cdot \vec{y} \in interior(A)$ for arbitrary $\vec{x}, \vec{y} \in A$ and for every $0 < t < 1$.

Lemma 3. The maps $h_+, h_- : \mathbb{R} \to [0, 2]$ are monotone increasing, respectively decreasing. In the case that the unit ball $\mathbf{B}$ of $(X, \|..\|)$ is strictly convex, then h_+, h_- are strictly monotone increasing, respectively decreasing.

Proof. See the tricky proof in [2],p.201, theorem 2.4 and p.202, theorem 2.5 . The authors only dealt with the map h_- . Note that they only prove that h_- is monotone. The case of a strictly convex unit ball $\mathbf{B}$ is not explicitly written down. You have to read both proofs of the theorems 2.4 and 2.5 attentively. $\square$

Remark 4. If we have a unit ball $\mathbf{B}$ of $(X, \|..\|)$ that is not strictly convex, then h_+, h_- are generally not strictly monotone. This is shown by the example of the normed space $(\mathbb{R}^2, \|..\|_\infty)$ with the norm $\|(x_1, x_2)\|_\infty := \max\{|x_1|, |x_2|\}$. Choose $\vec{x} := (1, 0)$, and $\vec{y} := (0, 1)$, then the interval in which h_+ is constant $(= 1)$ is $[-1, 0]$, while h_- is constant $(= 1)$ in $[0, 1]$. This example shows that the intervals where h_+ and h_-, respectively, are constant may intersect in one point. We are just proving the fact that both intervals do not intersect in an interval with nonempty interior.

Corollary 2. In the case that the unit ball $\mathbf{B}$ of a real normed space $(X, \|..\|)$ is strictly convex, then (An 11) is satisfied.

Proof. We had defined $\Theta(t) = \arccos\left(\frac{1}{4} \cdot \left[[h_+(t)]^2 - [h_-(t)]^2\right]\right)$, and because $\mathbf{B}$ is strictly convex it follows by the last lemma that h_+, h_- are strictly monotone increasing, respectively decreasing. $\square$

Now we want to prove that Θ remains to be strictly monotone decreasing, even if the unit ball is not strictly convex. Because h_-, h_+ are monotone, Θ is always monotone decreasing. We have to prove that the monotony is strict.
We prefer a direct proof. Assume that $-\infty < t_1 \le t_2 < +\infty$, and $\Theta(t_1) = \Theta(t_2)$.
The case $t_1 = t_2$ is possible. We will show that this is the only possible case. So let us assume that $-\infty < t_1 < t_2 < +\infty$. Now we hunt for contradictions. Because of $\Theta(t_1) = \Theta(t_2)$ and because h_+ is monotone increasing and h_- is monotone decreasing, we have that $h_+(t_1) = h_+(t_2)$ and $h_-(t_1) = h_-(t_2)$, and for all t in the interval $[t_1, t_2]$ both Θ and h_+ and h_- remain constant.
For the rest of the proof we calculate in coordinates of the basis $\{\frac{\vec{x}}{\|\vec{x}\|}, \frac{\vec{y}}{\|\vec{y}\|}\} = \{\mathsf{sign}(\vec{x}), \mathsf{sign}(\vec{y})\}$.
Hence $\mathsf{sign}(\vec{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathsf{sign}(\vec{y}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\vec{v} := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \mathsf{sign}(\vec{y} + t_1 \cdot \vec{x})$, $\vec{w} := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ $:= \mathsf{sign}(\vec{y} + t_2 \cdot \vec{x})$. We have $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\vec{y}+t_1\cdot\vec{x}}{\|\vec{y}+t_1\cdot\vec{x}\|} = \frac{t_1\cdot\|\vec{x}\|}{\|\vec{y}+t_1\cdot\vec{x}\|} \cdot \frac{\vec{x}}{\|\vec{x}\|} + \frac{\|\vec{y}\|}{\|\vec{y}+t_1\cdot\vec{x}\|} \cdot \frac{\vec{y}}{\|\vec{y}\|}$, hence $v_2 = \frac{\|\vec{y}\|}{\|\vec{y}+t_1\cdot\vec{x}\|} > 0$. For the same reason we get $w_2 = \frac{\|\vec{y}\|}{\|\vec{y}+t_2\cdot\vec{x}\|} > 0$.

Lemma 4. We have $\vec{v} \neq \vec{w}$.

Proof. We assume the opposite $\vec{v} = \vec{w}$. Because of $v_2 = w_2$, we have $v_2 = \frac{\|\vec{y}\|}{\|\vec{y}+t_1\cdot\vec{x}\|} = w_2 = \frac{\|\vec{y}\|}{\|\vec{y}+t_2\cdot\vec{x}\|}$, hence $\|\vec{y} + t_1 \cdot \vec{x}\| = \|\vec{y} + t_2 \cdot \vec{x}\|$. Then follows with $v_1 = w_1$ that $t_1 \cdot \|\vec{x}\| = t_2 \cdot \|\vec{x}\|$, hence $t_1 = t_2$. $\square$

Because of $h_+(t_1) = h_+(t_2)$ and $h_-(t_1) = h_-(t_2)$, we have that $h_+(t_1) = \left\| \begin{pmatrix} v_1 + 1 \\ v_2 \end{pmatrix} \right\|$ $= h_+(t_2) = \left\| \begin{pmatrix} w_1 + 1 \\ w_2 \end{pmatrix} \right\|$ and $h_-(t_1) = \left\| \begin{pmatrix} v_1 - 1 \\ v_2 \end{pmatrix} \right\| = h_-(t_2) = \left\| \begin{pmatrix} w_1 - 1 \\ w_2 \end{pmatrix} \right\|$.

For further investigations we must distinguish a few cases.

- Case (**A**): $v_1 < w_1$, with the subcases
  - Case (**A1**): $v_1 < w_1 < -1$ or $+1 < v_1 < w_1$.
  - Case (**A2**): $v_1 < w_1$ and $\{v_1, w_1\} \cap [-1, +1] \neq \emptyset$
- Case (**B**): $v_1 > w_1$, with the corresponding subcases (**B1**),(**B2**) .
- Case (**C**): $v_1 = w_1$, with the subcases
  - Case (**C1**): $v_1 = w_1 \in \{-1, +1\}$,
  - Case (**C2**): $v_1 = w_1 \in (-1, +1)$,
  - Case (**C3**): $v_1 = w_1 < -1$ or $v_1 = w_1 > +1$.

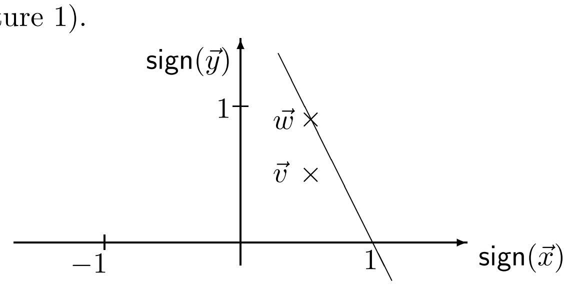
Let's start with the easy Case (**C1**): Assume, for instance, $v_1 = w_1 = 1$. Because of $h_-(t_1) = h_-(t_2)$ it follows $\left\| \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ w_2 \end{pmatrix} \right\|$, hence $v_2 = w_2$. This contradicts $\vec{v} \neq \vec{w}$. Or more detailed, $v_2 = w_2$ means $\|\vec{y} + t_1 \cdot \vec{x}\| = \|\vec{y} + t_2 \cdot \vec{x}\|$, hence $\frac{t_1 \cdot \|\vec{x}\|}{\|\vec{y}+t_1 \cdot \vec{x}\|} = v_1 = 1 = w_1 = \frac{t_2 \cdot \|\vec{x}\|}{\|\vec{y}+t_2 \cdot \vec{x}\|}$, which is only possible for $t_1 = t_2$, and there is a contradiction.

For all further cases, note that we can replace t_1, t_2 by $\widetilde{t_1}, \widetilde{t_2}$, with $t_1 < \widetilde{t_1} < \widetilde{t_2} < t_2$ and $\vec{v} := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \frac{\vec{y}+\widetilde{t_1} \cdot \vec{x}}{\|\vec{y}+\widetilde{t_1} \cdot \vec{x}\|}$, $\vec{w} := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} := \frac{\vec{y}+\widetilde{t_2} \cdot \vec{x}}{\|\vec{y}+\widetilde{t_2} \cdot \vec{x}\|}$ to make sure that all seven unit vectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \mathsf{sign}\begin{pmatrix} v_1 - 1 \\ v_2 \end{pmatrix}, \mathsf{sign}\begin{pmatrix} v_1 + 1 \\ v_2 \end{pmatrix}, \mathsf{sign}\begin{pmatrix} w_1 - 1 \\ w_2 \end{pmatrix}, \mathsf{sign}\begin{pmatrix} w_1 + 1 \\ w_2 \end{pmatrix}$$

are distinct.

Now we deal with the even more easier Case (**C2**): Thus $-1 < v_1 = w_1 < 1$. Because of $\vec{v} \neq \vec{w}$ we have $v_2 \neq w_2$, for instance $0 < v_2 < w_2$. Assume $0 \leq v_1 = w_1 < 1$. Hence, by convexity of the unit ball $\mathbf{B}$ of $(X, \|..\|)$, and $\vec{w}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbf{S}$, the straight line between both points is in $\mathbf{B}$. But then $\vec{v}$ would be in the interior of $\mathbf{B}$. This is impossible, because $\|\vec{v}\| = 1$, (see Picture 1).



Picture 1

Now we need a break. Let us turn our attention to 'Grecian Geometry', with this phrase we mean elementary geometry on the two-dimensional euclidean plane. We describe and prove two propositions about 'projections', which will be needed for the proof of the theorem.
For a better understanding of the following propositions it is reasonable to look at the enclosed drawings.

Proposition 2. Let us take $\mathbb{R}^2 = \{(\mathtt{x}|\mathtt{y})|\ \mathtt{x},\mathtt{y} \in \mathbb{R}\}$, the two-dimensional euclidean plane, with the horizontal x-axis and the vertical y-axis. Consider the two parallel lines $G_S : \mathtt{y} = \mathtt{x} - 1$ and $G_T : \mathtt{y} = \mathtt{x} + 1$. Assume a third straight line L , not parallel to G_S, G_T, respectively, with the property that L does not meet the origin $(0|0)$. The intersection of L with G_S is called $S = (\mathtt{x}_S|\mathtt{x}_S - 1)$, and the intersection of L with G_T is called $T = (\mathtt{x}_T|\mathtt{x}_T + 1)$.
Then there is an unique point $P_{hor} = (\mathtt{x}_{hor}|\mathtt{y}_{hor})$ on L , such that the three points $(0|0), (\mathtt{x}_{hor}+1|\mathtt{y}_{hor})$, and S are collinear, and such that the three points $(0|0), (\mathtt{x}_{hor}-1|\mathtt{y}_{hor})$, and T are collinear. (See the enclosed Picture 2).

Proof. In the case of $\mathtt{x}_S \neq \mathtt{x}_T$, we have an equation $L : \mathtt{y} = m_L \cdot \mathtt{x} + b_L,\ m_L, b_L \in \mathbb{R},\ b_L \neq 0,\ m_L \neq 1,$ and then take

$$\begin{aligned} P_{hor} = (\ \mathtt{x}_{hor}|\mathtt{y}_{hor}) &= \left(\frac{m_L - {b_L}^2}{b_L \cdot (m_L - 1)} \mid \frac{{m_L}^2 - {b_L}^2}{b_L \cdot (m_L - 1)} \right) \\ &= \left(\frac{\mathtt{x}_T \cdot (b_L + 1) + 1}{b_L} \mid m_L \cdot \frac{\mathtt{x}_T \cdot (b_L + 1) + 1}{b_L} + b_L \right) \\ &= \left(\frac{\mathtt{x}_S \cdot (b_L - 1) + 1}{b_L} \mid m_L \cdot \frac{\mathtt{x}_S \cdot (b_L - 1) + 1}{b_L} + b_L \right) \quad , \end{aligned}$$

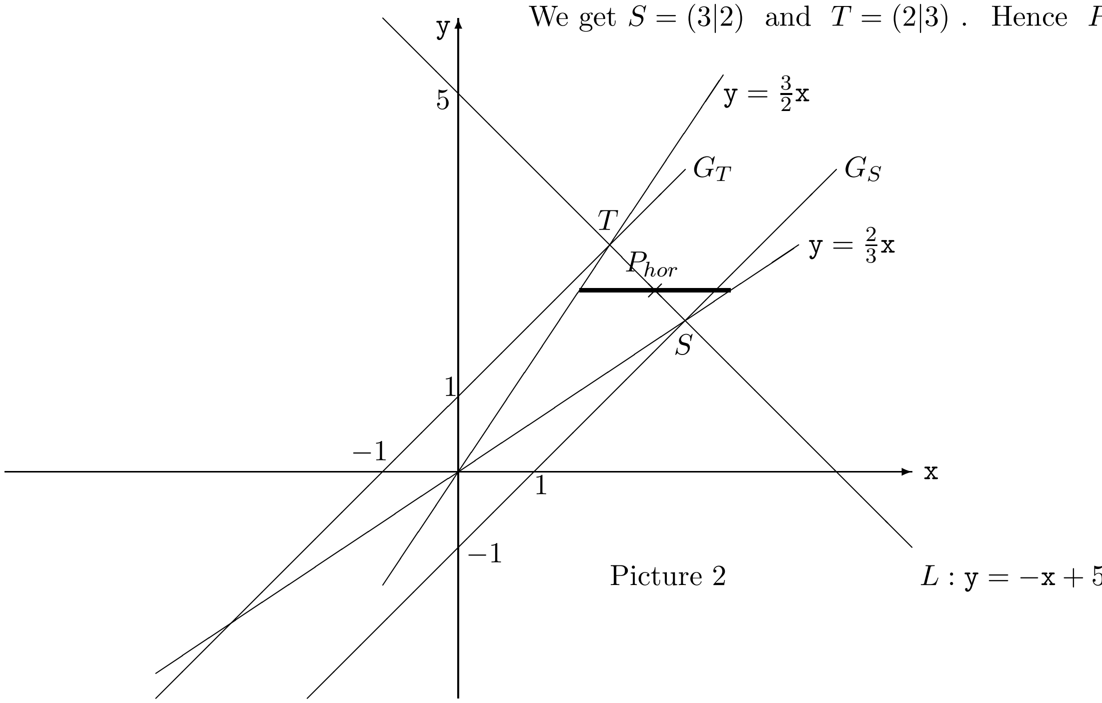
with $m_L = \frac{2+\mathtt{x}_T-\mathtt{x}_S}{\mathtt{x}_T-\mathtt{x}_S}$, and $b_L = \mathtt{x}_S \cdot (1 - m_L) - 1 = \mathtt{x}_T \cdot (1 - m_L) + 1$, and some elementary calculations confirm that indeed all three formulas of P_{hor} yield the same values for $\mathtt{x}_{hor}$ and $\mathtt{y}_{hor}$, and that P_{hor} fulfils the demanded properties. (See Picture 2) .
In the case of $\mathtt{x}_S = \mathtt{x}_T$, we have an equation $L :\ \mathtt{x} = a_L := \mathtt{x}_S = \mathtt{x}_T$, and we get

$$P_{hor} = (\ \mathtt{x}_{hor}|\mathtt{y}_{hor}) = \left(a_L \mid a_L - \frac{1}{a_L} \right) \quad , \quad \text{(see Picture 7).}$$

∎

Here: $m_L = -1,\ b_L = 5$.

We get $S = (3|2)$ and $T = (2|3)$. Hence $P_{hor} = (\frac{13}{5}|\frac{12}{5})$.

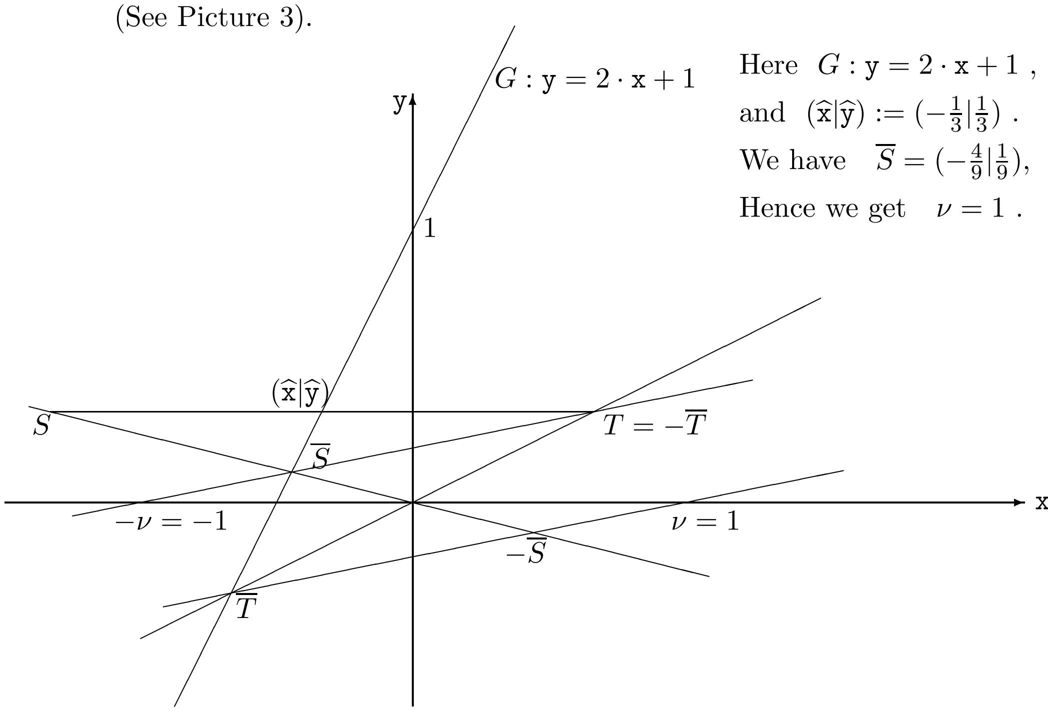


Picture 2

Proposition 3. Let us again take $\mathbb{R}^2$ with the horizontal x-axis and the vertical y-axis. Consider the straight line G with the equation $\mathtt{y} = m \cdot \mathtt{x} + 1,\ m \in \mathbb{R}\backslash\{-1,+1\}$. Let us choose an arbitrary point $(\widehat{\mathtt{x}}|\widehat{\mathtt{y}})$ on G, $\widehat{\mathtt{y}} \neq 0$. Take two points $S := (\widehat{\mathtt{x}}-1|\widehat{\mathtt{y}})$ and $T := (\widehat{\mathtt{x}}+1|\widehat{\mathtt{y}})$. We call $\overline{S}$ the projection of S on the line G , and $\overline{T}$ the projection of T on the line G. (That means that the three points $(0|0), S, \overline{S}$, and the three points $(0|0), T, \overline{T}$, respectively, are collinear, $\overline{S}, \overline{T} \in G$.) The four points $\overline{S}, \overline{T}, -\overline{S}, -\overline{T}$ are the corners of a parallelogram. We claim that the intersection ν of the line that connects $\overline{T}$ and $-\overline{S}$ with the horizontal x-axis has the value 1, and the intersection of the line that connects $-\overline{T}$ and $\overline{S}$ with the x-axis is -1 . Moreover, the two points $\overline{S}, \overline{T}$ are on the same side of the horizontal x-axis if and only if $-1 < m < 1$.

(See Picture 3).

Here $G : \mathtt{y} = 2 \cdot \mathtt{x} + 1$, hence $m = 2$,
and $(\widehat{\mathtt{x}}|\widehat{\mathtt{y}}) := (-\frac{1}{3}|\frac{1}{3})$.
We have $\overline{S} = (-\frac{4}{9}|\frac{1}{9})$, $\overline{T} = (-\frac{2}{3}| - \frac{1}{3})$.
Hence we get $\nu = 1$.



Picture 3

Proof. With elementary calculations, we have that

$$\overline{S} \ = \ \frac{1}{1+m} \cdot (\widehat{\mathtt{x}} - 1 \mid \widehat{\mathtt{y}}) \qquad \text{and} \qquad \overline{T} \ = \ \frac{1}{1-m} \cdot (\widehat{\mathtt{x}} + 1 \mid \widehat{\mathtt{y}}) \ .$$

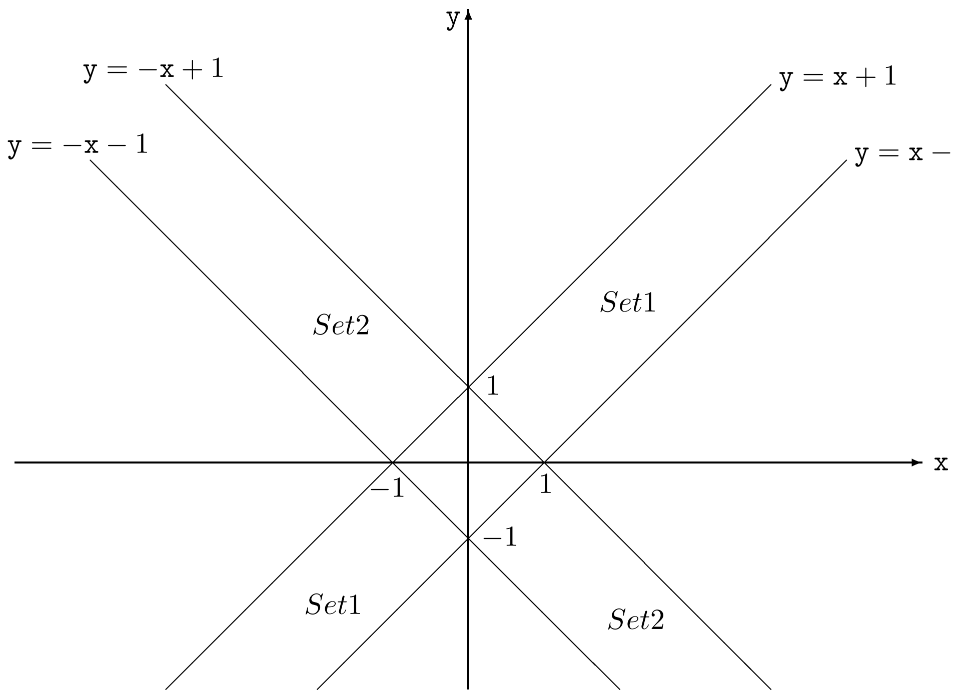
Some more calculations yield the formula $\mathtt{y} = \frac{m \cdot \widehat{\mathtt{x}} + 1}{\widehat{\mathtt{x}} + m} \cdot [\ \mathtt{x} - 1\ ]$ for the straight line that intersects $\overline{T}$ and $-\overline{\mathrm{S}}$, and finally we get $\nu = 1$.
And $\overline{S}, \overline{T}$ are on the same side of the horizontal $\mathtt{x}$-axis if and only if their second components have the same signs, hence if and only if $(1-m) \cdot (1+m) > 0$. □

Before we can return to our main purpose, (that is to prove **Theorem 1**) , we have to mention one easy fact about unit balls in the $\mathbb{R}^2$.

Lemma 5. Assume that $\mathbb{R}^2$ is provided with a seminorm $\|..\|$ with the unit ball $\mathbf{B}$. Define two closed sets $Set1$ and $Set2$,
$Set1 := \{(\mathtt{x}|\mathtt{y}) \in \mathbb{R}^2 \quad | \quad \mathtt{x}+1 \geq \mathtt{y} \geq \mathtt{x}-1\}$, $Set2 := \{(\mathtt{x}|\mathtt{y}) \in \mathbb{R}^2 \quad | -\mathtt{x}+1 \geq \mathtt{y} \geq -\mathtt{x}-1\}$.

Assume now that we have the four unit vectors $(1|0),(-1|0),(0|1),(0|-1)$, that means $\|(1|0)\| = \|(-1|0)\| = \|(0|1)\| = \|(0|-1)\| = 1$.
Then we have that $\mathbf{B} \subset Set1 \cup Set2$.

Proof. Instead of a proof we prefer to show a picture and we remark that $\mathbf{B}$ has to be convex. (See Picture 4). ☐



Picture 4

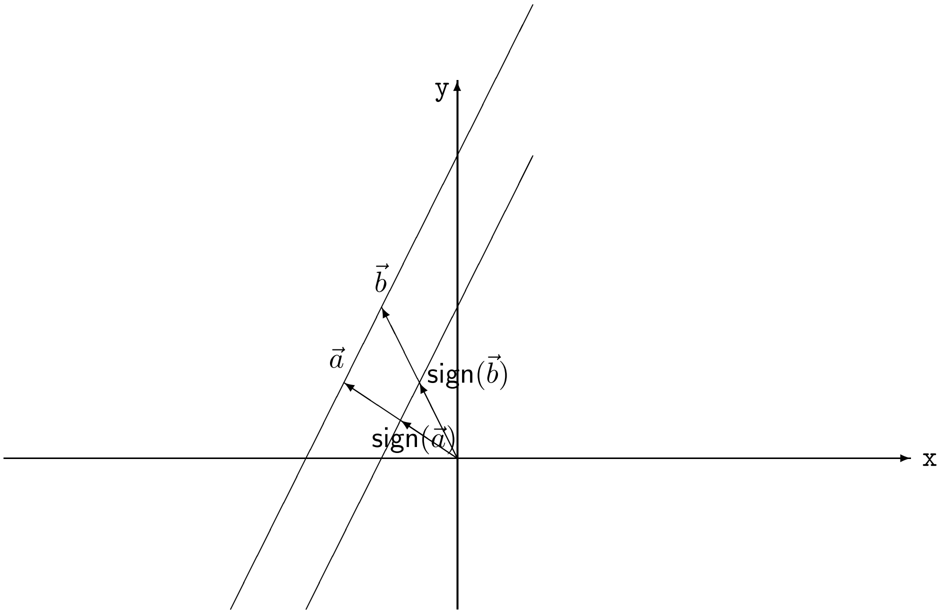
The time has come to return to the proof of **Theorem 1** , but we still need some general preparations. Recall that we had two unit vectors

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\vec{y}+t_1\cdot\vec{x}}{\|\vec{y}+t_1\cdot\vec{x}\|} \neq \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{\vec{y}+t_2\cdot\vec{x}}{\|\vec{y}+t_2\cdot\vec{x}\|}$, with $t_1 < t_2$, and $v_2, w_2 > 0$.

Because they are different, they uniquely determine a straight line L that connects them. If we define two lines L_- and L_+ such that L_- connects $\vec{v} - (1|0) = (v_1 - 1|v_2)$ and $\vec{w} - (1|0) = (w_1 - 1|w_2)$, and such that L_+ connects $(v_1 + 1|v_2)$ and $(w_1 + 1|w_2)$, it is trivial that all the three lines L , L_- , and L_+ are parallel, (see Picture 6) .

Lemma 6. Assume an arbitrary **hw space** $(Y, \|..\|)$ with a homogeneous weight $\|..\|$. Let $\vec{a}, \vec{b} \in Y$ be linear independent, and let $\|\vec{a}\| = \|\vec{b}\| > 0$. Consider the two-dimensional subspace of Y , generated by the vectors $\vec{a}, \vec{b}$, which is isomorphic to the vector space $\mathbb{R}^2$.
Then the line that connects $\vec{a}$ and $\vec{b}$ is parallel to the line that connects sign($\vec{a}$) and sign($\vec{b}$).

Proof. Trivial by the intercept theorem and the fact that $\|..\|$ is homogeneous. (See Picture 5) . ☐



Picture 5

Further we define two straight lines $L_{-,\mathsf{sign}}$ and $L_{+,\mathsf{sign}}$, such that $L_{-,\mathsf{sign}}$ connects $\mathsf{sign}(v_1 - 1|v_2)$ and $\mathsf{sign}(w_1 - 1|w_2)$, and such that $L_{+,\mathsf{sign}}$ connects $\mathsf{sign}(v_1 + 1|v_2)$ and $\mathsf{sign}(w_1 + 1|w_2)$.

Lemma 7. We claim that all the five lines $L,\ L_-,\ L_+,\ L_{-,\mathsf{sign}}$, and $L_{+,\mathsf{sign}}$ are parallel.

Proof. On page 9 we described the conditions that we have $h_+(t_1) = \left\| \begin{pmatrix} v_1 + 1 \\ v_2 \end{pmatrix} \right\| = h_+(t_2) = \left\| \begin{pmatrix} w_1 + 1 \\ w_2 \end{pmatrix} \right\|$ and $h_-(t_1) = \left\| \begin{pmatrix} v_1 - 1 \\ v_2 \end{pmatrix} \right\| = h_-(t_2) = \left\| \begin{pmatrix} w_1 - 1 \\ w_2 \end{pmatrix} \right\|$,
and all norms are greater than 0, hence together with the previous lemma the claim is true. $\square$

Lemma 8. Recall that on page 9 we had defined six unit vectors

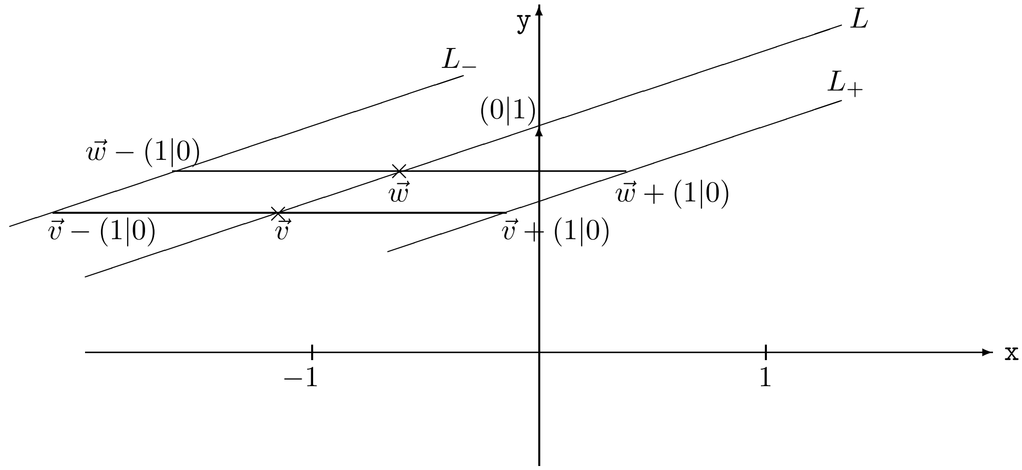
$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},\ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},\ \mathsf{sign}\begin{pmatrix} v_1 - 1 \\ v_2 \end{pmatrix},\ \mathsf{sign}\begin{pmatrix} v_1 + 1 \\ v_2 \end{pmatrix},\ \mathsf{sign}\begin{pmatrix} w_1 - 1 \\ w_2 \end{pmatrix},\ \mathsf{sign}\begin{pmatrix} w_1 + 1 \\ w_2 \end{pmatrix},$$

and we mentioned on page 10 that, without loss of generality, all these vectors are distinct, and they are different from $(0|1)$. Then all the six points are collinear.
In the case that these six vectors are on different sides of the vertical y-axis, even all seven vectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},\ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},\ \mathsf{sign}\begin{pmatrix} v_1 - 1 \\ v_2 \end{pmatrix},\ \mathsf{sign}\begin{pmatrix} v_1 + 1 \\ v_2 \end{pmatrix},\ \mathsf{sign}\begin{pmatrix} w_1 - 1 \\ w_2 \end{pmatrix},\ \mathsf{sign}\begin{pmatrix} w_1 + 1 \\ w_2 \end{pmatrix}$$

are collinear.

Proof. All these seven unit vectors have a positive second component, thus they are above the horizontal x-axis. By the previous lemma, the three lines $L,\ L_{-,\mathsf{sign}}$, and $L_{+,\mathsf{sign}}$ are parallel. Let us consider, for instance, L and $L_{-,\mathsf{sign}}$. L meets $\vec{v}$ and $\vec{v}$, and $L_{-,\mathsf{sign}}$ meets $\mathsf{sign}(v_1 - 1|v_2)$ and $\mathsf{sign}(w_1 - 1|w_2)$. Because of the convexity of the unit ball $\mathbf{B}$, the case $L \neq L_{-,\mathsf{sign}}$ is not possible. Hence all six points that generate these three lines have to be on the same straight line $L = L_{-,\mathsf{sign}} = L_{+,\mathsf{sign}}$. If in addition these six vectors are on different sides of the vertical y-axis, the unit vector $(0|1)$ has to be on the same line L, too .
(See Picture 6). $\square$



Picture 6

Now we have collected all the facts we will need in the following. Recall that we wanted to prove **Theorem 1** , and that we already have proved Case (**C1**) and Case (**C2**). Still there are missing the cases **C3**, **A1**, **A2**, **B1**, and **B2** .
Recall that we calculate with the basis $\{\frac{\vec{x}}{\|\vec{x}\|}, \frac{\vec{y}}{\|\vec{y}\|}\} = \{\mathsf{sign}(\vec{x}), \mathsf{sign}(\vec{y})\}$, and that we have $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\vec{y}+t_1\cdot\vec{x}}{\|\vec{y}+t_1\cdot\vec{x}\|} \neq \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{\vec{y}+t_2\cdot\vec{x}}{\|\vec{y}+t_2\cdot\vec{x}\|}$, with $t_1 < t_2$, and $v_2, w_2 > 0$.
Because of $\vec{v} \neq \vec{w}$ there is an unique straight line L that connects both points, with an equation $L : \mathtt{y} = m_L \cdot \mathtt{x} + b_L$, $m_L, b_L \in \mathbb{R}$ or $L : \mathtt{x} = a_L$, (if $v_1 = w_1 =: a_L$) .

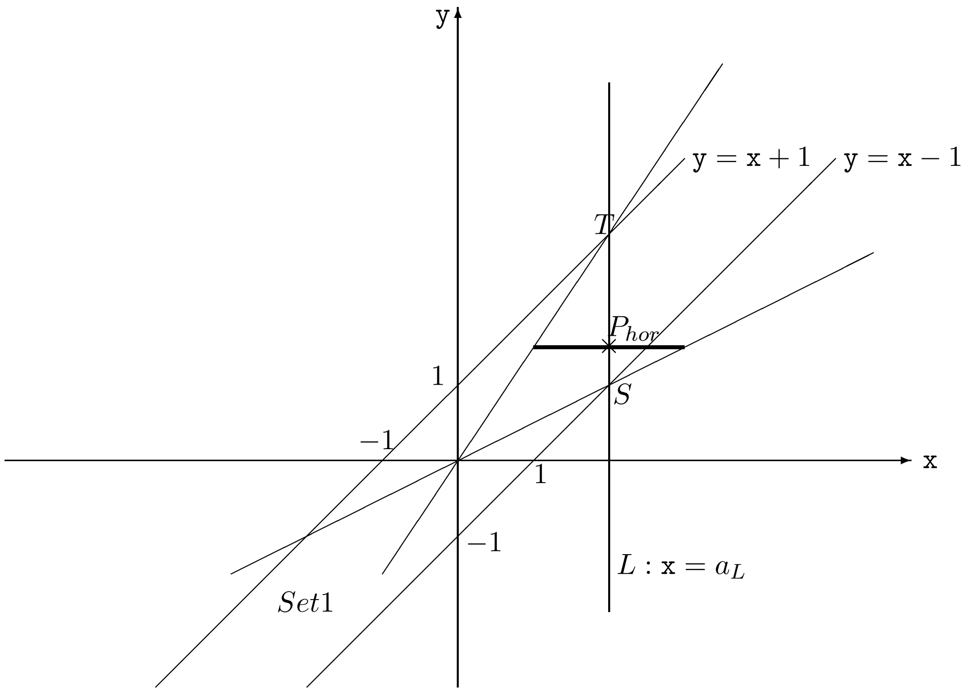
Case (**C3**): $v_1 = w_1 < -1$ or $+1 < v_1 = w_1$, for instance, assume $1 < v_1 = w_1$. Thus L is vertical, $L : \mathtt{x} = a_L := v_1 = w_1$.
Thus we deduce with proposition 2 that there is an unique point P_{hor} , (see Picture 7) ,

$$P_{hor} = (\, \mathtt{x}_{hor}|\mathtt{y}_{hor}) = \left(a_L \mid a_L - \frac{1}{a_L}\right) ,$$

such that the three points $(0|0), (a_L - 1|a_L - \frac{1}{a_L})$, and $T := (a_L|a_L + 1)$ are collinear, and such that the three points $(0|0), (a_L + 1|a_L - \frac{1}{a_L})$, and $S := (a_L|a_L - 1)$ are collinear. Because of $a_L > 1$, P_{hor} is between S and T. By the last lemma 8, all six unit vectors

$$\vec{v},\ \vec{w},\ \mathsf{sign}(\vec{v} + (1|0)),\ \mathsf{sign}(\vec{w} + (1|0)),\ \mathsf{sign}(\vec{v} - (1|0)),\ \mathsf{sign}(\vec{w} - (1|0))$$

are located on the same line L, and they have to be in the set $Set1$ (by lemma 5), and above the $\mathtt{x}$-axis. Now we have to use proposition 2 . Assume that $\vec{v} := P_{hor}$.
If we have a $\vec{w}$ from the point P_{hor} on L upward, the point $\mathsf{sign}(\vec{w} - (1|0))$ will be above T , hence not in the set $Set1$. If we have a $\vec{w}$ from P_{hor} on L downward, $\mathsf{sign}(\vec{w} + (1|0))$ will be below S, hence not in $Set1$. Thus the only possibility is $P_{hor} = \vec{v} = \vec{w}$, (and $\mathsf{sign}(\vec{v} + (1|0)) = \mathsf{sign}(\vec{w} + (1|0)) = S$ and $\mathsf{sign}(\vec{v} - (1|0)) = \mathsf{sign}(\vec{w} - (1|0)) = T$) , and we find a contradiction, and Case (**C3**) is discussed.



Picture 7

The next Case (**A1**) will be proved in a similar way.
We have $v_1 < w_1 < -1$ or $+1 < v_1 < w_1$, assume $1 < v_1 < w_1$. The two different points $\vec{v}, \vec{w}$ are in the set $Set1$, and they define an unique straight line $L : \mathtt{y} = m_L \cdot \mathtt{x} + b_L$, $m_L, b_L \in \mathbb{R}$. If $m_L = 0$ we have $b_L \geq 1$ (otherwise, if $0 < b_L < 1$, it contradicts the convexity of the unit ball **B**, note the unit vectors $(0|1), \vec{v}, \vec{w}$). If $m_L \neq 0$, we have a zero $a_L := -b_L/m_L$ of L. It must be both $1 \leq |a_L|$ and $1 \leq |b_L|$. In all other cases, namely $-1 < a_L < 1$ or $-1 < b_L < 1$, we have a contradiction to the convexity of **B**. (Note the six unit vectors $(1|0), (-1|0), (0|1), (0|-1), \vec{v}, \vec{w}$). Hence, $1 \leq |b_L|$ and, if $m_L \neq 0$, $1 \leq |a_L|$.

Lemma 9. Furthermore, we have $m_L \neq 1$.

Proof. Assume $m_L = 1$. If $\vec{v}, \vec{w} \in interior(Set1)$, it contradicts the convexity of **B**, (note $(0|1), (1|0), \vec{v}, \vec{w}$). Or, if $\vec{v}, \vec{w} \in L : \mathtt{y} = \mathtt{x} + 1$ or $\vec{v}, \vec{w} \in L : \mathtt{y} = \mathtt{x} - 1$, with $\|\vec{v} + (1|0)\| = \|\vec{w} + (1|0)\|$ or $\|\vec{v} - (1|0)\| = \|\vec{w} - (1|0)\|$ always follows $\vec{v} = \vec{w}$. □

We call $S := (\mathtt{x}_S|\mathtt{x}_S - 1)$ the intersection of L and $\mathtt{y} = \mathtt{x} - 1$, and $T := (\mathtt{x}_T|\mathtt{x}_T + 1)$ the intersection of L and $\mathtt{y} = \mathtt{x} + 1$, $S, T \in Set1$. Then, by proposition 2, we have an unique point $P_{hor} = (\mathtt{x}_{hor}|\mathtt{y}_{hor})$ on L , such that the three points $(0|0), (\mathtt{x}_{hor} + 1|\mathtt{y}_{hor})$, and S are collinear, and such that the three points $(0|0), (\mathtt{x}_{hor} - 1|\mathtt{y}_{hor})$, and T are collinear, (see again Picture 2 for an example) , and P_{hor} has the representation

$$P_{hor} \ = \ (\ \mathtt{x}_{hor}|\mathtt{y}_{hor}) = \left(\frac{m_L - {b_L}^2}{b_L \cdot (m_L - 1)} \ | \ \frac{{m_L}^2 - {b_L}^2}{b_L \cdot (m_L - 1)} \right) .$$

By lemma 5 and lemma 8 , all six unit vectors

$$\vec{v}, \ \vec{w}, \ \mathsf{sign}(\vec{v} + (1|0)), \ \mathsf{sign}(\vec{w} + (1|0)), \ \mathsf{sign}(\vec{v} - (1|0)), \ \mathsf{sign}(\vec{w} - (1|0))$$

are located on the same line L, and they have to be elements of $Set1$. The point $\vec{v}$ could be P_{hor} . Then $S = \mathsf{sign}(\vec{v} + (1|0))$ and $T = \mathsf{sign}(\vec{v} - (1|0))$. If we imagine that $\vec{w}$ is located away from $\vec{v}$ in one direction on L or the other, either $\mathsf{sign}(\vec{w} + (1|0))$ is not in $Set1$ or

$\mathsf{sign}(\vec{w}-(1|0))$ is not in $Set1$. Hence it is only possible that $\vec{v}=\vec{w}=P_{hor}$, which contradicts our assumption. Thus we have proved Case (**A1**).

Finally follows the last case (**A2**), because if we can prove this, the other cases (**B1**) and (**B2**) can be shown in the same manner, and no other ideas are needed. Hence we only prove Case (**A2**). Let $v_1 < w_1$ and $\{v_1, w_1\} \cap [-1,+1] \neq \emptyset$, for instance let $-1 \leq w_1 \leq 1$. The two different points $\vec{v}, \vec{w}$ define an unique straight line $L : \mathtt{y} = m_L \cdot \mathtt{x} + b_L$, $m_L, b_L \in \mathbb{R}$. By lemma 8 all seven points $(0|1), \vec{v},\ \vec{w},\ \mathsf{sign}(\vec{v}+(1|0)),\ \mathsf{sign}(\vec{w}+(1|0)),\ \mathsf{sign}(\vec{v}-(1|0))$, $\mathsf{sign}(\vec{w}-(1|0))$ are located on L, hence $b_L = 1$.

Lemma 10. We have that $-1 < m_L < 1$.

Proof. If $m_L < -1$ or $m_L > +1$, it would contradict the convexity of the unit ball $\mathbf{B}$. (Note the four unit vectors $(-1|0), \vec{w}, \vec{v}, (1|0)$). Now we assume $m_L \in \{-1, 1\}$, for instance $m_L = 1$. Then $\vec{w}, \vec{v}$ are on $L : \mathtt{y} = \mathtt{x} + 1$, hence the three vectors $(0|0), \vec{v}+(1|0), \vec{w}+(1|0)$ would be collinear on the line $\mathtt{y} = \mathtt{x}$. Because of $\|\vec{v}+(1|0)\| = h_+(t_1) = h_+(t_2) = \|\vec{w}+(1|0)\|$ it follows that $\vec{v}+(1|0) = \vec{w}+(1|0)$, hence $\vec{v}=\vec{w}$, and we get a contradiction. □

Now we use proposition 3 . Abbreviate $\overline{T}_w := \mathsf{sign}(\vec{w}+(1|0))$, $\overline{S}_w := \mathsf{sign}(\vec{w}-(1|0))$, $\overline{T}_v := \mathsf{sign}(\vec{v}+(1|0))$, and $\overline{S}_v := \mathsf{sign}(\vec{v}-(1|0))$. Now consider the eight unit vectors $\overline{T}_w, \overline{S}_w, \overline{T}_v, \overline{S}_v$, and their negatives $-\overline{T}_w, -\overline{S}_w, -\overline{T}_v, -\overline{S}_v$. Four at a time create a parallelogram, namely $\overline{T}_w, \overline{S}_w, -\overline{T}_w, -\overline{S}_w$, and $\overline{T}_v, \overline{S}_v, -\overline{T}_v, -\overline{S}_v$, respectively. By proposition 3 , the line that connects $\overline{T}_v$ and $-\overline{S}_v$ and also the line that connects $\overline{T}_w$ and $-\overline{S}_w$ meet the horizontal $\mathtt{x}$-axis in the point $(1|0)$, and, because of $-1 < m_L < 1$, by proposition 3 the two points $\overline{T}_v, \overline{S}_v$ and $\overline{T}_w, \overline{S}_w$, respectively, both are located above the horizontal axis, hence the intersection point $(1|0)$ is between $\overline{T}_v$, $-\overline{S}_v$ and $\overline{T}_w$, $-\overline{S}_w$, respectively. Now let us consider the line J that connects the unit vectors $\overline{T}_w$ and $-\overline{S}_v$. Because of the convexity of $\mathbf{B}$, J must be a subset of $\mathbf{B}$. Because of our assumption $v_1 < w_1$, on the line L the most left one of the six different unit vectors $\overline{S}_v, \overline{S}_w, \vec{v}, \vec{w}, \overline{T}_v$ and $\overline{T}_w$ is $\overline{S}_v$, and the most right one of the six is $\overline{T}_w$. Correspondingly, of the six unit vectors $-\overline{S}_v, -\overline{S}_w, -\vec{v}, -\vec{w}, -\overline{T}_v, -\overline{T}_w$, the most left one of the six is $-\overline{T}_w$, and the most right one is $-\overline{S}_v$. We take on the lines L and $-L$, respectively, the points that are the most right ones, namely $\overline{T}_w$ and $-\overline{S}_v$. The line J that connects both points $\overline{T}_w$ and $-\overline{S}_v$ crosses the $\mathtt{x}$-axis in $\lambda > 1$. Because of $J \subset \mathbf{B}$, for the norm of $(\lambda|0)$ holds that $\|(\lambda|0)\| \leq 1$. Because of $\lambda > 1$, $(1|0)$ would be in the interior of $\mathbf{B}$. That contradicts $\|(1|0)\| = 1$, and finally we have found a contradiction also for the last Case (**A2**) .
That means that all cases (**A1**), (**A2**), (**B1**), (**B2**), (**C1**), (**C2**), and (**C3**) have been discussed, and with the assumption '$t_1 < t_2$' , that means '$\vec{v} \neq \vec{w}$' , we always found a contradiction, thus $\vec{v} = \vec{w}$ and $t_1 = t_2$ remains as the only possibility. Hence the map Θ is injective, hence bijective, and finally **Theorem 1** has been proved ! □

6 On Concave Corners and Some Open Problems

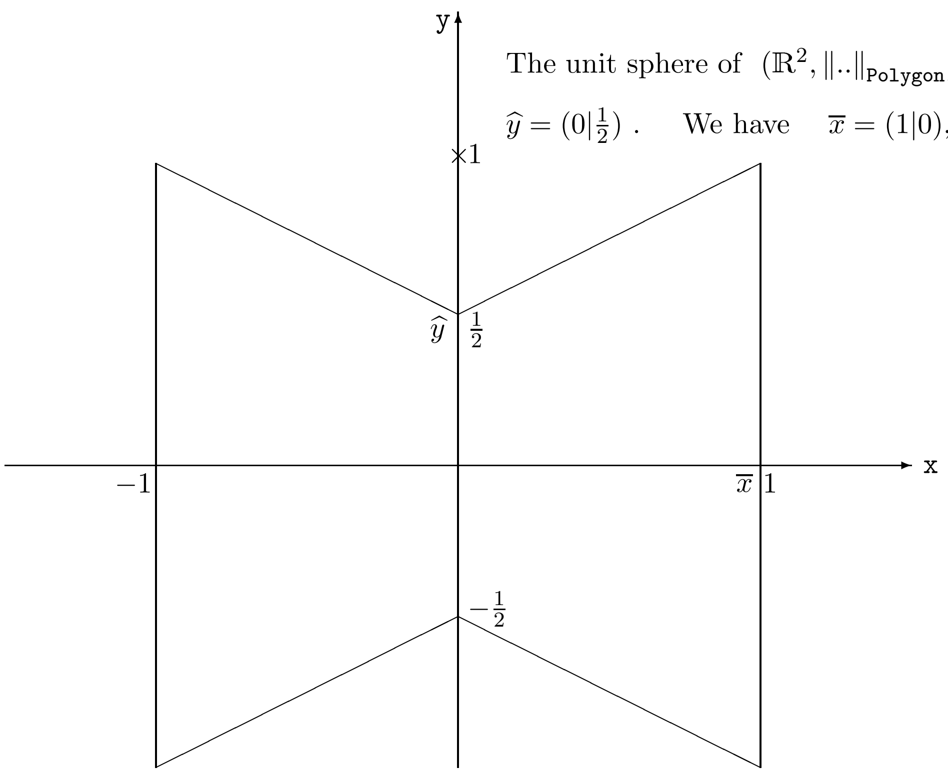
In proposition 1 we claimed and proved that in a real seminormed vector space $(X, \|..\|)$ the triple $(X, \|..\|, <..|..>_\spadesuit)$ fulfils the CSB inequality, hence the 'Thy angle' $\angle_{Thy}(\vec{x}, \vec{y})$ is defined for all $\vec{x}, \vec{y} \neq \vec{0}$. Furthermore, in **Theorem 1** we proved that if $(X, \|..\|)$ even is a normed space, the axiom (`An 11`) is fulfiled. For proving both facts we always use the convexity of the unit ball $\mathbf{B}$ of $(X, \|..\|)$. If you note that the convexity of the unit ball is equivalent to the triangle inequality, we come to the natural question whether there exists a **hw space** $(X, \|..\|)$, such that $\|..\|$ fulfils (1), the absolute homogenity, and (2), the positive

definiteness, but not (3), the triangle inequality, and $(X, \|..\|, < .. \mid .. >_\spadesuit)$ satisfies the CSB inequality, or even the axiom (An 11). Natural candidates are the spaces $(\mathbb{R}^2, \|..\|_{\texttt{Polygon},r})$, which will be defined at once. But in the case of $r < 1$ we have no success, as we will see. For $r < 1$, $(\mathbb{R}^2, \|..\|_{\texttt{Polygon},r})$ has something that we now call a ' concave corner'.

Definition 3. Let the pair $(X, \|..\|)$ be a **hw space**, let $\widehat{y} \in X$, $\|\widehat{y}\| > 0$.
$\widehat{y}$ is called a *concave corner* $\iff$ there is an $\overline{x} \in X$, $\|\overline{x}\| > 0$, and there are three real numbers $\varepsilon,\ m_-,\ m_+$, with $m_- < m_+$ and $\varepsilon > 0$, such that for all $\delta \in [0, \varepsilon]$ we have that $\|\delta \cdot \mathsf{sign}(\overline{x}) + (1 + \delta \cdot m_+) \cdot \mathsf{sign}(\widehat{y})\| = 1 = \| - \delta \cdot \mathsf{sign}(\overline{x}) + (1 - \delta \cdot m_-) \cdot \mathsf{sign}(\widehat{y})\|$.

We get a set of homogeneous weights on $\mathbb{R}^2$ if we define for every $r > 0$ a homogeneous weight $\|..\|_{\texttt{Polygon},r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$, if we fix the unit sphere $\mathbf{S}$ of $(\mathbb{R}^2, \|..\|_{\texttt{Polygon},r})$ with the polygon through the six points $\{(0|r), (1|1), (1|-1), (0|-r), (-1|-1), (-1|1)\}$ and returning to $(0|r)$, and then extending $\|..\|_{\texttt{Polygon},r}$ by homogenity.
(See Picture 8) .

The unit sphere of $(\mathbb{R}^2, \|..\|_{\texttt{Polygon},\frac{1}{2}})$, with the concave corner $\widehat{y} = (0|\frac{1}{2})$. We have $\overline{x} = (1|0),\ \varepsilon = 1,\ m_- = -1 < m_+ = +1$.



Picture 8

Lemma 11. For all $0 < r < 1$, the space $(\mathbb{R}^2, \|..\|_{\texttt{Polygon},r})$ has a concave corner at $\widehat{y} := (0|r)$, with $\overline{x} := (1|0)$, $\varepsilon := 1$, $m_- := 1 - \frac{1}{r} < 0 < m_+ := \frac{1}{r} - 1$.

Proposition 4. Let the pair $(X, \|..\|)$ be a **hw space**, let $\widehat{y} \in X$ be a concave corner. Then $(X, \|..\|, < .. \mid .. >_\spadesuit)$ does not fulfil the CSB inequality.

Proof. We use the above elements $\widehat{y},\ \overline{x}$ and then for all $\delta \in [0, \varepsilon]$ we can compute

$$P_\spadesuit(\delta) := < \delta \cdot \mathsf{sign}(\overline{x}) + (1 + \delta \cdot m_+) \cdot \mathsf{sign}(\widehat{y}) \ \mid - \delta \cdot \mathsf{sign}(\overline{x}) + (1 - \delta \cdot m_-) \cdot \mathsf{sign}(\widehat{y}) >_\spadesuit$$

$$= \tfrac{1}{4} \cdot \Big[\ \| \ [2 + \delta \cdot (m_+ - m_-)] \cdot \mathsf{sign}(\widehat{y}) \ \|^2 \ - \ \| \ 2 \cdot \delta \cdot \mathsf{sign}(\overline{x}) + \delta \cdot (m_+ + m_-) \cdot \mathsf{sign}(\widehat{y}) \ \|^2 \ \Big]$$

$$= \tfrac{1}{4} \cdot \Big[\ [2 + \delta \cdot (m_+ - m_-)]^2 \cdot \| \ \mathsf{sign}(\widehat{y}) \ \|^2 \ - \ \delta^2 \cdot \| \ 2 \cdot \mathsf{sign}(\overline{x}) + (m_+ + m_-) \cdot \mathsf{sign}(\widehat{y}) \ \|^2 \ \Big]$$

$$= 1 + \delta \cdot (m_+ - m_-) + \tfrac{1}{4} \cdot \delta^2 \cdot \Big[\ (m_+ - m_-)^2 \ - \ \| \ 2 \cdot \mathsf{sign}(\overline{x}) + (m_+ + m_-) \cdot \mathsf{sign}(\widehat{y}) \ \|^2 \ \Big]$$

$= \quad 1+\delta\cdot(m_+-m_-)+\frac{1}{4}\cdot\delta^2\cdot K \quad ,$
with the real constant $\quad K := \ (m_+-m_-)^2 \ - \ \|\ 2\cdot\mathsf{sign}(\overline{x})+(m_++m_-)\cdot\mathsf{sign}(\widehat{y})\ \|^2$.
This calculation holds for all $\delta\in[0,\varepsilon]$. Hence, for a positive $\widehat{\delta}$ that is almost 0 , because of $m_+-m_->0$ we have $P_\spadesuit(\widehat{\delta})>1$, hence, for the unit vectors
$\widehat{\delta}\cdot\mathsf{sign}(\overline{x})+(1+\widehat{\delta}\cdot m_+)\cdot\mathsf{sign}(\widehat{y})$ and $-\widehat{\delta}\cdot\mathsf{sign}(\overline{x})+(1-\widehat{\delta}\cdot m_-)\cdot\mathsf{sign}(\widehat{y})$,
the CSB inequality is not satisfied. □

Corollary 3. For all $0<r<1$, the space $(\mathbb{R}^2,\|..\|_{\texttt{Polygon},r},<..\,|\,..>_\spadesuit)$ does not fulfil the CSB inequality. Hence, there are vectors $\vec{x}\neq\vec{0}\neq\vec{y}$ such that the 'Thy angle' $\angle_{Thy}(\vec{x},\vec{y})$ is not defined. Hence, for $0<r<1$, the space $(\mathbb{R}^2,\|..\|_{\texttt{Polygon},r},\angle_{Thy})$ is not an 'angle space' .

As we mentioned in the beginning of the section, we formulate two open problems. We always deal with pseudonormed vector spaces $(X,\|..\|)$ (that are **hw spaces** which are positive definite, that means $\|\vec{x}\|=0$ only for $\vec{x}=\vec{0}$) with a non-convex unit ball.
Problem 1: Exists a pseudonormed vector space $(X,\|..\|)$, such that $\|..\|$ does not fulfil the triangle inequality, but $(X,\|..\|,<..\,|\,..>_\spadesuit)$ satisfies the CSB inequality ?
Problem 2: If Problem 1 is true, exists a pseudonormed vector space $(X,\|..\|)$, such that $\|..\|$ does not fulfil the triangle inequality, but $(X,\|..\|,<..\,|\,..>_\spadesuit)$ satisfies the CSB inequality and the axiom (`An` 11) is fulfiled ?
Good candidates for both questions are the *Hölder weights* $\|..\|_p$ on $\mathbb{R}^2$, $\|(x,y)\|_p = \sqrt[p]{|x|^p+|y|^p}$, with $0<p<1$, but p almost 1 .

7 A Generalization of the Thy Angle

Recall in the general definitions the expression of a convex set, and for a subset A of $(X,\|..\|)$ we defined the convex hull *conv(A)* ,

$$conv(A) := \bigcup\{\sum_{i=1}^{n} t_i\cdot\vec{x_i}\mid n\in\mathbb{N},\ t_i\in[0,1] \text{ and } \vec{x_i}\in A \text{ for } i=1,...,n \text{ , and } \sum_{i=1}^{n} t_i=1\}\ ,$$

which is the smallest convex set that contains A.
For a **hw space** $(X,\|..\|)$ with the unit ball $\mathbf{B}:=\{\vec{x}\in X\mid\|\vec{x}\|\le 1\}$, we defined a seminorm $\|..\|_{|conv(\mathbf{B})}$, for all $\vec{x}\in X$, let $\|\vec{x}\|_{|conv(\mathbf{B})} := \ \inf\{r>0\mid\frac{1}{r}\cdot\vec{x}\in conv(\mathbf{B})\}$.
We have for all $\vec{x}\in X$ that $\|\vec{x}\|_{|conv(\mathbf{B})} \le \|\vec{x}\|$. Note that $(X,\|..\|)\xrightarrow{id}(X,\|..\|_{|conv(\mathbf{B})})$ is a continuous map. Further note that for a **hw space** $(X,\|..\|)$ with the unit ball $\mathbf{B}$, the pair $(X,\|..\|_{|conv(\mathbf{B})})$ is a seminormed vector space, because the triangle inequality is satisfied. Then we called $\|..\|$, or the pair ($X,\|..\|$), respectively, *normable* if and only if the pair $(X,\|..\|_{|conv(\mathbf{B})})$ is a normed vector space.
In a normable space $(X,\|..\|)$ we have the zero-set $\mathsf{Z}=\{\vec{0}\}$. Then there is an equivalence $\vec{x}\neq\vec{0}\iff\|\vec{x}\|>0\iff\|\vec{x}\|_{|conv(\mathbf{B})}>0$, for all $\vec{x}\in X$.

Definition 4. Let the pair $(X,\|..\|)$ be a normable **hw space**. We define a continuous product $<..\,|\,..>_{\spadesuit\ |conv}: X^2\longrightarrow\mathbb{R}$.
If $\vec{x}=\vec{0}$ or $\vec{y}=\vec{0}$ we set $<\vec{x}\,|\,\vec{y}>_{\spadesuit\ |conv} := 0$, and in the case of $\vec{x}\neq\vec{0}\neq\vec{y}$ we define $<\vec{x}\,|\,\vec{y}>_{\spadesuit\ |conv} :=$

$$\frac{1}{4}\cdot\|\vec{x}\|\cdot\|\vec{y}\|\cdot\left[\left\|\frac{\vec{x}}{\|\vec{x}\|_{|conv(\mathbf{B})}}+\frac{\vec{y}}{\|\vec{y}\|_{|conv(\mathbf{B})}}\right\|^2_{|conv(\mathbf{B})} - \left\|\frac{\vec{x}}{\|\vec{x}\|_{|conv(\mathbf{B})}}-\frac{\vec{y}}{\|\vec{y}\|_{|conv(\mathbf{B})}}\right\|^2_{|conv(\mathbf{B})}\right].$$

Proposition 5. Let $(X, \|..\|)$ be a normable **hw space** with the unit ball $\mathbf{B}$.
(1) The product $< ..|.. >_{\spadesuit\ |conv} : X^2 \longrightarrow \mathbb{R}$ fulfils the properties $\overline{(1)}$ ("homogenity"), $\overline{(2)}$ ("symmetry"), $\overline{(3)}$ ("positive semidefiniteness") , and $\overline{(4)}$ ("definiteness") . Hence, $(X, < ..|.. >_{\spadesuit\ |conv})$ is a homogeneous product vector space.
(2) We have $\|\vec{x}\| = \sqrt{< \vec{x} \,|\, \vec{x} >_{\spadesuit\ |conv}}$ for all $\vec{x} \in X$.
(3) The triple $(X, \|..\|, < ..|.. >_{\spadesuit\ |conv})$ fulfils the CSB inequality, that means for all $\vec{x}, \vec{y} \in X$ it holds that $| < \vec{x} \,|\, \vec{y} >_{\spadesuit\ |conv} | \ \leq \ \|\vec{x}\| \cdot \|\vec{y}\|$.
(4) In the case of a normed space $(X, \|..\|)$ we get that $\|..\| = \|..\|_{|conv(\mathbf{B})}$, and $< ..|.. >_{\spadesuit\ |conv} = < ..|.. >_{\spadesuit}$.

Proof. (1) See the comments after the definition of the product $< ..|.. >_{\spadesuit}$ on page 5 .
(2) Trivial.
(3) Easy, because $\|..\|_{|conv(\mathbf{B})}$ fulfils the triangle inequality.
(4) Trivial, because $\mathbf{B}$ is convex, hence $\mathbf{B} = conv(\mathbf{B})$. □

Definition 5. For all normable **hw spaces** $(X, \|..\|)$, for all $\vec{x}, \vec{y} \in X \backslash \mathsf{Z}$, (that means $\vec{x} \neq \vec{0}$ and $\vec{y} \neq \vec{0}$) with $| < \vec{x} \,|\, \vec{y} >_{\spadesuit\ |conv} | \leq \|\vec{x}\| \cdot \|\vec{y}\|$, we define the
generalized Thy angle $\angle_{\overline{\mathbf{Thy}}}(\vec{x}, \vec{y}) \ := \ \arccos \frac{< \vec{x} \,|\, \vec{y} >_{\spadesuit\ |conv}}{\|\vec{x}\| \cdot \|\vec{y}\|} \ =$

$$\arccos \left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|_{|conv(\mathbf{B})}} + \frac{\vec{y}}{\|\vec{y}\|_{|conv(\mathbf{B})}} \right\|^2_{|conv(\mathbf{B})} - \left\| \frac{\vec{x}}{\|\vec{x}\|_{|conv(\mathbf{B})}} - \frac{\vec{y}}{\|\vec{y}\|_{|conv(\mathbf{B})}} \right\|^2_{|conv(\mathbf{B})} \right] \right) .$$

Proposition 6. Let the pair $(X, \|..\|)$ be a normable **hw space**.
(a) The triple $(X, \|..\|, < .. \,|\, .. >_{\spadesuit\ |conv})$ fulfils the CSB inequality, hence the generalized angle $\angle_{\overline{\mathbf{Thy}}}(\vec{x}, \vec{y})$ is defined for all $\vec{x} \neq \vec{0} \neq \vec{y}$.
(b) The triple $(X, \|..\|, \angle_{\overline{\mathbf{Thy}}})$ fulfils all the demands (An 1), (An 2), (An 3), (An 4), (An 5), (An 6), (An 7) . Hence $(X, \|..\|, \angle_{\overline{\mathbf{Thy}}})$ is an angle space.
(c) If even the pair $(X, \|..\|)$ is a normed vector space, then we have for all $\vec{x}, \vec{y} \neq \vec{0}$ that $\angle_{\overline{\mathbf{Thy}}}(\vec{x}, \vec{y}) \ = \ \angle_{Thy}(\vec{x}, \vec{y})$.
(d) If even $(X, < .. \,|\, .. >_{IP})$ is an **IP space**, then we have for all $\vec{x}, \vec{y} \neq \vec{0}$ that $\angle_{\overline{\mathbf{Thy}}}(\vec{x}, \vec{y}) \ = \ \angle_{Euclid}(\vec{x}, \vec{y})$.
(e) If $(X, \|..\|)$ is a normable **hw space**, then the triple $(X, \|..\|, \angle_{\overline{\mathbf{Thy}}})$ generally does not fulfil (An 8), (An 9), (An 10) .
(f) If $(X, \|..\|)$ is a normable **hw space**, then $(X, \|..\|, \angle_{\overline{\mathbf{Thy}}})$ satifies (An 11) .

Proof. (a) The CSB inequality was shown in the previous proposition.
(b) Easy. See the proofs in proposition 1.
(c) Trivial, because $\mathbf{B} = conv(\mathbf{B})$, hence $\|..\| = \|..\|_{|conv(\mathbf{B})}$.
(d) Trivial, because in an **IP space** is $\angle_{Euclid} = \angle_{Thy}$.
(e) Use the examples of proposition 1.
(f) The proof is not trivial, but the same as in **Theorem 1** . □

Acknowledgements
We wish to thank Dr. Gencho Skordev and Prof. Dr. Eberhard Oeljeklaus , which supported us by patient listening and interested discussions.

References

[1] Karl Menger , " Some Applications of Point-Set Methods " , Annals of Mathematics 32, 1931, 739-750

[2] Charles Dimminie, Edward Andalafte, Raymond Freese , " Angles in Normed Linear Spaces and a Characterization of Real Inner Product Spaces ", Mathematische Nachrichten, 129 (1986) 197-204

[3] Charles Dimminie, Edward Andalafte, Raymond Freese , " Generalized Angles and a Characterization of Inner Product Spaces ", Houston Journal of Mathemaics 14, 1988, No.4, 457-480

[4] J.E.Valentine, S.G.Wayment , " Wilson Angles in Linear Normed Spaces ", Pacific Journal of Mathematics, Vol 36 , No 1, 1971, 239-243

[5] J.E.Valentine, C.Martin , " Angles in Metric and Normed Linear Spaces", Colloquium Mathematicum, Vol 34, 1976, 209-217

[6] Carlos Benitez , " Orthogonality in Normed Linear Spaces: a classification of the Different Concepts and Some Open Problems", Revista Matematica de la Universidad Madrid, Vol 2, 1989, 53-57

[7] S. Gudder, D. Strawther , " Orthogonally Additive and Orthogonally Increasing Functions on Vector Spaces", Pacific Journal of Mathematics, Vol 58 , No 2, 1975, 427-436

[8] Ivan Singer , " Unghiuri Abstracte si Functii Trigonometrice in Spatii Banach", Buletin Stiintific, Sectia de Stiinte Matematice si Fisice, Academia Republicii Populare Romîne, Vol 9, (1957), 29-42

[9] Pavle M. Miličić , " Sur le g-angle dans un espace normé ", Matematik Vesnik, 45(1993), 43-48

[10] Pavle M. Miličić , " Characterizations of Convexities of Normed Spaces by Means of g-Angles", Matematik Vesnik, 54(2002), 37-44

[11] Pavle M. Miličić , " On the B-Angle and g-Angle in Normed Spaces ", Journal of Inequalities in Pure and Applied Mathematics, vol.8, iss.4, (2007), 1-18

[12] H. Gunawan, J. Lindiarni, O. Neswan , " P-, I-, g-, and D-Angles in Normed Spaces ", Institute of Technology Bandung(Indonesia), J. Sci., Vol.40 A, No.1, (2008), 24-32

[13] Reinhold Meise, Dietmar Vogt , " Introduction to Functional Analysis ", Oxford University Press 1997

[14] Gert K. Pedersen , " Analysis Now " , Springer 1989

[15] Horst Martini, Konrad J. Swanepoel, Gunter Weiß , " The Geometry of Minkowski Spaces - A Survey. Part 1 ", EXPOSITIONES MATHEMATICAE 19 (2001) 97-142